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ON A CONJECTURE OF BARRY SIMON ON TRACE IDEALS

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1. Setting of the problem. Let $(\Omega, \mathscr{F}, \mu)$ be any measure space and let A and B be two (bounded linear) operators on $L_2 = L_2(\Omega, \mathscr{F}, \mu)$. We say that B *pointwise dominates* A if $|Ax(t)| \leqq B|x|(t)$, μ-a.e., for all $x \in L_2$ (see [5, def. on p. 36]).

This definition also makes sense in arbitrary Banach function lattices X over $(\Omega, \mathscr{F}, \mu)$. In fact, the definition is connected with the fundamental notion of regular operators in general Banach lattices; see, e.g., the monograph by Schaefer [4].

A special instance of pointwise domination is the following: Let T_K be the integral operator generated by a $\mu \otimes \mu$-measurable kernel $K: \Omega \times \Omega \to \mathbb{R}$ (or $\mathbb{C}$), i.e.,

$$T_K x(s) = \int_\Omega K(s, t)x(t)\, d\mu(t), \qquad x \in X, \qquad s \in \Omega.$$

Then clearly $T_{|K|}$ pointwise dominates T_K.

In his lecture notes "Trace Ideals and Their Applications," Barry Simon [5] considered pointwise domination for operators belonging to trace ideals (Schatten p-classes) S_p. Let us briefly recall the definition of these classes.

Given any operator T on a Hilbert space H, the *singular numbers* (or singular values) of T are $s_n(T) := \inf\{\|T - T_n\|: \text{rank } T_n < n\}$. For compact operators T we have $s_n(T) = \lambda_n(|T|)$, where $|T| = (T^*T)^{1/2}$, and the λ_n's are the nonzero eigenvalues of $|T|$, arranged in nonincreasing order and repeated according to their algebraic multiplicities (see, e.g., [2] or [5]).

Clearly, the sequence of singular numbers is nonincreasing—$\|T\| = s_1(T) \geqq s_2(T) \geqq \cdots \geqq 0$—and, moreover, $s_n(UTV) \leqq \|U\| s_n(T) \|V\|$, $n = 1, 2, \ldots$, holds for any Hilbert space operators T, U, V.

The *Schatten p-classes* $S_p = S_p(H)$ consist of all operators on H with the following property:

$$(s_n(T)) \in \begin{cases} \ell_p & \text{if } 0 < p < \infty \\ c_0 & \text{if } p = \infty. \end{cases}$$

These classes reflect compactness properties of operators. Among them, the most important ones are S_1, S_2, and S_∞, the nuclear, Hilbert-Schmidt, and compact operators, respectively.

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