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HEIGHTS FOR LOCAL SYSTEMS ON CURVES

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0. Introduction. In this article, we develop a theory of heights for a "motivic" local system V on a dense open set X^* of a smooth, projective algebraic curve X over a global field k. Such a theory was suggested by P. Deligne, on the basis of the work of B. Gross and D. Zagier [13]. V manifests itself in ℓ-adic realizations V_ℓ (an ℓ-adic local system on X^*) for all ℓ, as well as a polarized variation of Hodge structures for each infinite place. We consider 0-cycles on X^* with coefficients in V, of the type $\Sigma_x[v_x]$, where v_x is an "absolute Hodge-Tate cycle" in the fiber of V at x (see §4 for these notions). For each place $\mathscr{P}$, we have a pairing $\langle v_1, v_2\rangle_{\mathscr{P}}$ (the local height pairing at $\mathscr{P}$), defined whenever v_1 and v_2 have disjoint supports. This definition is rather involved, and not entirely satisfactory at ramified places. The problem there is that one uses ℓ-adic cohomology, so some assumptions are needed so that the numbers obtained are rational, and independent on ℓ.

The construction itself follows Beilinson's [2] construction of height pairings as linking numbers in various cohomology theories: at infinite places, Beilinson-Deligne cohomology with coefficients in V [1], obtained, in this case, from the L_2-complex of Zucker [23] and its Hodge filtration; at finite places, étale cohomology (over a nonalgebraically closed field). Beilinson [1], [2], has explained most convincingly why these two theories are entirely analogous. In many ways the definitions of the height pairing at infinite and finite places are parallel.

A special feature of the archimedean case is the expression of the (local) height in terms of Green's functions with values in V. Such Green functions are harmonic in the sense of [23] and have the usual log-type singularities along the diagonal.

We explain, in an appendix, the relation of this theory to the higher-dimensional heights of Beilinson [2]. If $Y \xrightarrow{\pi} X$ is a projective morphism with Y smooth and π smooth over X^*, and if $Z_1 \subset \pi^{-1}(x_1)$, $Z_2 \subset \pi^{-1}(x_2)$ are algebraic cycles such that $d_1 + d_2 = \dim(Y) + 1$, where d_i is the codimension of Z_i in Y, then (when $x_1 \neq x_2$) one has Beilinson's height $\langle Z_1, Z_2\rangle$. On the other hand, the cohomology class of V_i gives an absolute Hodge-Tate cycle v_i in the stalk at x_i of some local system V_i. One has a bilinear pairing $V_1 \times V_2 \to \mathbb{Q}$, hence a height $\langle v_1, v_2\rangle$ is defined. The result is that $\langle Z_1, Z_2\rangle = \langle v_1, v_2\rangle$.

This shows one reason to develop a theory of heights for cycles with coefficients in local systems: to compute higher-dimensional heights, for instance, in Kuga fiber varieties, by the geometry of a local system on a curve. As initially observed by Deligne, the local height, for some cycles over Heegner points of modular curves $X_0(N)$, allows one to express Gross-Zagier's results on derivatives, at the center of

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