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NONEXISTENCE OF HOMOTOPY FORMULA FOR (0, 1) FORMS ON HYPERSURFACES IN $\mathbb{C}^3$

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This note is about the following, vaguely stated problem, which arises in the question of the embeddability of *CR* structures of hypersurface type of real dimension 5 (cf. [1], [6], [8]):

Given g a (0, 1) form defined on a strictly convex hypersurface in $\mathbb{C}^3$, and assuming that $\bar{\partial}_b g \neq 0$ but that $\bar{\partial}_b g$ is small (in some suitable sense), one attempts to solve approximately, locally, the equation $\bar{\partial}_b u = g$.

We observe that this is not possible without some control on g itself. Although this is not conclusive for the applications in view, this is enough to preclude the possibility of a "homotopy formula."

1. Nonexistence of homotopy formula. In $\mathbb{C}^n$ homotopy formulas exist for solving $\bar{\partial}_b$, locally on strictly convex hypersurfaces, for $(0, q)$ forms of degree q, $1 \leqslant q \leqslant n-3$; cf. [3]. For $(0, n-2)$ forms one can still locally solve the equation $\bar{\partial}_b u = g$ if g is a $\bar{\partial}_b$ closed $(0, n-2)$ form, but the proof given in [3], pp. 89–92, requires a special trick which does not lead to a homotopy formula.

Let us concentrate on the case of (0, 1) forms on hypersurfaces in $\mathbb{C}^3$. By a *homotopy formula* one means a formula of the type

$$\omega = \bar{\partial}_b(P\omega) + Q(\bar{\partial}_b\omega).$$

It would have the effect that if ω were a (0, 1) form such that $\bar{\partial}_b\omega$ was small in some reasonable sense, then one should be able to find a function u such that the difference $(\omega - \partial_b u)$ was "small" (just set $u = P\omega$).

This goes against the following fact:

THEOREM. *Let S_5 be the unit sphere in $\mathbb{C}^3$. Let Σ' and Σ be nonempty open subsets of S_5 such that $\Sigma' \subset \Sigma$, $\bar{\Sigma} \neq S_5$. There exists a sequence (g_j) of smooth (0, 1) forms defined on Σ such that $\bar{\partial}_b g_j$ tends to 0 in the $\mathscr{C}^\infty$ topology, but such that for every smooth function u defined on Σ',*

$$\sup_{\Sigma'} |\bar{\partial}_b u - g_j| \geqslant 1.$$

Proof. By using Moebius transformations, we can assume, without loss of generality, that Σ' contains the intersection of S_5 with the complex hyperplane

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