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# CLOSED GEODESICS IN HOMOLOGY CLASSES ON SURFACES OF VARIABLE NEGATIVE CURVATURE

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**0. Introduction.** Let $M$ be a compact Riemannian manifold of negative curvature. It is well known that there exist countably many closed geodesics in $M$, and that if $N(t)$ is the number of such closed geodesics with lengths $\leqslant t$ then

$$N(t) \sim e^{ht}/ht \tag{0.1}$$

as $t \to \infty$, where $h > 0$ is the topological entropy of the flow [9]. Recently, Phillips and Sarnak [12] and Katsuda and Sunada [6] have investigated the asymptotic behavior (as $t \to \infty$) of $N(t; m)$, the number of closed geodesics in the homology class $m$ with lengths $\leqslant t$. For manifolds $M$ with *constant* negative curvature they prove that for each homology class $m$, as $t \to \infty$

$$N(t; m) \sim Ce^{ht}/t^{1+r/2} \tag{0.2}$$

for a constant $C > 0$, where $r$ is the rank (over $\mathbb{Z}$) of the homology group $H_1M$ (i.e., $H_1M \cong \mathbb{Z}^r \oplus G$, where $G$ is the torsion subgroup).

The purpose of this paper is to extend (0.2) to manifolds of *variable* negative curvature, and to describe the asymptotics of $N(t; m)$ when $m$ varies with $t$ in a roughly linear fashion. For simplicity we shall only consider surfaces $M$ whose first homology groups are torsion free, i.e., $H_1M \cong \mathbb{Z}^{2g}$, $g \geqslant 2$. There exist $C^\infty$ forms $\omega_1, \ldots, \omega_{2g}$ on $M$ such that for any smooth closed curve $\gamma$ on $M$ the homology class of $\gamma$ is $(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_{2g})$. Let $SM$ be the unit tangent bundle of $M$; define $W_i: SM \to \mathbb{R}$ by $W_i(x, v) = \langle \omega_i(x), v \rangle$ (here $<, >$ denotes dot product). For $\xi \in \mathbb{R}^{2g}$ define $-\Gamma(\xi)$ to be the maximum entropy of an invariant probability measure $\lambda$ on $SM$ satisfying $\int W_i \, d\lambda = \xi_i \forall i = 1, 2, \ldots, 2g$ (invariant means invariant with respect to the geodesic flow on $SM$). In sec. 4 we will show that $-\Gamma(\xi)$ is well defined and $C^\infty$ for $\xi$ in some neighborhood of the origin, and that the Hessian matrix $\nabla^2\Gamma(\xi)$ is strictly positive definite for every $\xi$ in this neighborhood.

The main result of this paper is

THEOREM 1. *Let $\xi = t^{-1}(m_1, \ldots, m_{2g})$; then as $t \to \infty$*

$$N(t; m) \sim e^{-t\Gamma(\xi)}t^{-g-1}(2\pi)^{-g}(\det \nabla^2\Gamma(\xi))^{1/2}(\langle \nabla\Gamma(\xi), \xi \rangle - \Gamma(\xi))^{-1} \tag{0.3}$$

*uniformly for $\xi$ in some neighborhood of the origin.*

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