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ON REIDER'S METHOD AND HIGHER ORDER EMBEDDINGS

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Introduction. Let L be a numerically effective and big line bundle on a smooth projective surface S. Questions about the spannedness and very ampleness of $K_S \otimes L$ arise naturally; cf. [Bom], [So-V]. Recently, Reider [Rdr] introduced a technique which yields answers to these questions that are not obtainable by previous methods.

Motivated by Bombieri's classical work [Bom], we are interested in using Reider's method to answer the following

QUESTION. *Let S be a smooth projective surface on which K_S is ample (this is relaxed to nef and big in §2). What is the smallest integer $t > 0$ so that the map associated to $\Gamma(K_S^t)$ gives a "kth order-embedding?"*

The first problem is to decide what we mean by a kth-order embedding. We introduce the concept of k-spannedness. Let $\mathscr{L}$ be a line bundle on S (resp. on a nonsingular curve C). We say that $\mathscr{L}$ is *k-spanned* for $k \geqslant 0$ if for any distinct points $z_1, \ldots, z_r$ on S (resp. on C) and any positive integers $k_1, \ldots, k_r$ with $\Sigma_{i=1}^r k_i = k + 1$, the map

$$\Gamma(\mathscr{L}) \to \Gamma(\mathscr{L} \otimes \mathcal{O}_{\mathscr{Z}})$$

is onto, where $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$ is an (admissible) 0-cycle defined by the ideal sheaf $\mathscr{I}_{\mathscr{Z}}$, where $\mathscr{I}_{\mathscr{Z}}\mathcal{O}_{S,z}$ is isomorphic to $\mathcal{O}_{S,z}$ (respectively $\mathcal{O}_{C,z}$) for $z \notin \{z_1, \ldots, z_r\}$ and $\mathscr{I}_{\mathscr{Z}}\mathcal{O}_{S,z_i}$ is generated by $(x_i, y_i^{k_i})$ at z_i, with (x_i, y_i) local coordinates at z_i on S, $i = 1, \ldots, r$ (respectively $\mathscr{I}_{\mathscr{Z}}$ is generated by $y_i^{k_i}$, y_i local coordinate at z_i on C).

Note that 0-spannedness is equivalent to $\mathscr{L}$ being spanned and 1-spannedness is equivalent to $\mathscr{L}$ being very ample.

There are a number of other notions of kth-order embedding (see §3 for some discussion). Our choice of the above definition was guided by two criteria: (1) the definition should be the weakest definition that includes the obvious examples (e.g., if L is very ample, then L^k should give a kth-order embedding) but for which strong results can be proven; (2) there should be a strong criterion for $K_S \otimes L$ to give a kth-order embedding, where L is nef and big.

In this article we show that there is a very satisfactory answer to (2). We use this to answer the question for which positive t the line bundle K_S^t (respectively K_S^{-t}) is k-spanned where K_S (respectively K_S^{-1}) is ample. In [Be-So] a detailed investigation of k-spannedness is made.

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