Vol. 58, No. 2 DUKE MATHEMATICAL JOURNAL © April 1989

FAMILIES OF WEIERSTRASS POINTS

FERNANDO CUKIERMAN

Introduction. Let $\bar{\mathcal{M}}_g$ and $\bar{\mathcal{C}}_g$ be the coarse moduli spaces of stable and pointed stable algebraic curves of genus g over the complex numbers, and $\pi: \bar{\mathcal{C}}_g \to \bar{\mathcal{M}}_g$ the natural map. The $\mathbb{Q}$-vector space $\mathrm{Pic}(\bar{\mathcal{C}}_g) \otimes \mathbb{Q}$ has a basis

$$\{\omega, \lambda, [\Delta_0], \ldots, [\Delta_{g-1}]\},$$

where $\omega = \omega_\pi$ is the relative dualizing sheaf of π, $\lambda = \pi^* \det \pi_* \omega$, and Δ_i are the boundary components of $\bar{\mathcal{C}}_g$. If $\bar{\mathcal{W}} \subset \bar{\mathcal{C}}_g$ is the closure of the locus of Weierstrass points of smooth curves, then $\bar{\mathcal{W}}$ is linearly equivalent to a linear combination $a\omega + b\lambda + \Sigma\, c_i[\Delta_i]$.

In section 2 we determine the coefficients $a, b, c_0, \ldots, c_{g-1}$: In a one-parameter family of smooth curves degenerating to a reducible curve with one node, the Wronskian determinant constructed from regular 1-forms vanishes along the components of the central fiber; the essential point of our computation is determining these orders of vanishing.

In section 3 we study some of the geometry of $\bar{\mathcal{W}}$, namely, its singularities and intersection with the boundary of $\bar{\mathcal{C}}_g$. Along the interior $\mathcal{C}_g$, the divisor $\mathcal{W}$ behaves like a generic determinantal variety, at least in the sense that it is irreducible and singular only at points corresponding to Weierstrass points of weight at least two. Also, $\bar{\mathcal{W}}$ is generically transverse to Δ_i, for $i > 0$, but this generic behavior fails along Δ_0: $\bar{\mathcal{W}}$ is singular in codimension one along the locus $N \subset \Delta_0$ of nodes of irreducible curves. The singularity of $\bar{\mathcal{W}}$ along N consists of several hypercuspidal branches, coming from singularities of the Hurwitz scheme parametrizing $\bar{\mathcal{W}}$.

Let $\bar{\mathcal{D}} \subset \bar{\mathcal{M}}_g$ (resp. $\bar{\mathcal{E}} \subset \bar{\mathcal{M}}_g$) be the closure of the divisor of smooth curves C possessing a point p such that $(g-1)p$ moves in a pencil (resp. $(g+1)p$ moves in a net). The closure of the locus of smooth curves possessing a Weierstrass point of weight at least two is $\bar{\mathcal{D}} \cup \bar{\mathcal{E}}$, and the divisors $\bar{\mathcal{D}}$ and $\bar{\mathcal{E}}$ then form the branch divisor of $\pi: \bar{\mathcal{W}} \to \bar{\mathcal{M}}_g$.

In section 5 we apply to π the Hurwitz's formula of section 4, taking into account the contribution of $\Delta_0 \subset \bar{\mathcal{M}}_g$ due to the singularity of $\bar{\mathcal{W}}$ along N, to obtain a relation among the classes of $\bar{\mathcal{D}}$ and $\bar{\mathcal{E}}$. Combining this with the determination by S. Diaz [D1] of the class of $\bar{\mathcal{D}}$, we obtain a formula for the class of $\bar{\mathcal{E}}$.

The main tools for understanding what happens to Weierstrass points when smooth curves degenerate are the theories of admissible covers [HM] and of limit

Received May 2, 1988. Revision received July 25, 1988.