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SMOOTH SOLUTIONS OF DEGENERATE LAPLACIANS ON STRICTLY PSEUDOCONVEX DOMAINS

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Introduction. A fundamental feature in the theory of the Dirichlet problem for a uniformly elliptic operator is C^∞ regularity up to the boundary: if the data are infinitely differentiable, then the solution is infinitely differentiable up to the boundary. However, if the operator is merely elliptic in the interior of the domain but degenerates at the boundary, then C^∞ regularity at the boundary may fail. This paper studies some ramifications of this phenomenon for a class of degenerate elliptic operators that arise in several complex variables.

If Ω is a smoothly bounded connected strictly pseudoconvex domain in $\mathbf{C}^{n+1}$ with defining function $\varphi < 0$ in Ω, then near $b\Omega, \log(-1/\varphi)$ is strictly plurisubharmonic, so that $(i/2)\partial\bar{\partial}\log(-1/\varphi)$ defines a Kähler metric. The operators of interest are the Laplace-Beltrami operators for these metrics. These Laplacians have coefficients which are smooth at the boundary, but the ellipticity degenerates there. Their prototype is the Bergman Laplacian on the ball $\mathbf{B}^{n+1} = \{z\colon |z|^2 < 1\} \subset \mathbf{C}^{n+1}$, obtained by taking $\varphi = |z|^2 - 1$ and given by

$$\Delta_0 = 4(1 - |z|^2)\sum(\delta^{jk} - z^j\bar{z}^k)\frac{\partial^2}{\partial z^j\,\partial\bar{z}^k}.$$

This operator was studied in [G1], where it was shown that despite the fact that the Dirichlet problem $\Delta_0 u = 0$, $u|_{b\mathbf{B}^{n+1}} = f$ is solvable for arbitrary continuous boundary data f, in order that $u \in C^\infty(\overline{\mathbf{B}^{n+1}})$ it is necessary and sufficient that $f \in C^\infty(b\mathbf{B}^{n+1})$ be the boundary value of a pluriharmonic function, and thus that u be pluriharmonic.

Our main results deal with the analogous question of characterizing the global smooth solutions of $\Delta_\varphi u = 0$ on a general strictly pseudoconvex domain, where Δ_φ is the Laplacian for the metric with Kähler form $(i/2)\partial\bar{\partial}\log(-1/\varphi)$. Interestingly, the case $n = 1$ differs from $n \geqslant 2$. When $n = 1$ we have derived a sufficient condition on Ω and φ that all smooth global solutions of $\Delta_\varphi u = 0$ be pluriharmonic. We say that Ω has a *transverse symmetry* if there is a one-parameter family of biholomorphisms of $\bar{\Omega}$ whose infinitesimal generator is everywhere transverse to the complex tangent space $H(b\Omega) = T(b\Omega) \cap JT(b\Omega)$ at $b\Omega$. (Here J is the almost complex structure map on $\mathbf{C}^{n+1}$.) For example, a *circular domain*

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