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DERIVATIONS OF VON NEUMANN ALGEBRAS INTO THE COMPACT IDEAL SPACE OF A SEMIFINITE ALGEBRA

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1. Introduction and statement of results. Let M be a semifinite von Neumann algebra and let $\mathscr{J}(M)$ be the norm closed two-sided ideal generated by the finite projections of M. Let $N \subseteq M$ be a subalgebra of M. A derivation of N into $\mathscr{J}(M)$ is a linear application δ: $N \mapsto \mathscr{J}(M)$ satisfying $\delta(xy) = \delta(x)y + x\delta(y)$ for $x, y \in N$. For instance, if $K \in \mathscr{J}(M)$, then the derivation $\delta(x) = (\operatorname{ad} K)(x) = Kx - xK$ is of this type. Such derivations implemented by elements in $\mathscr{J}(M)$ are called *inner*. There are many examples of derivations of *-subalgebras $N \subseteq M$ into the ideal $\mathscr{J}(M)$ which are not inner. A typical such example is as follows: Take $M = \mathscr{B}(\mathrm{L}^2(\mathbb{T}, \mu))$, where μ is the Lebesgue measure on the torus $\mathbb{T}$, let $N = \mathrm{C}(\mathbb{T})$ act on $\mathrm{L}^2(\mathbb{T}, \mu)$ by left multiplication, and define $\delta(x) = (\operatorname{ad} \mathrm{P}_{\mathrm{H}^2})(x)$, where $\mathrm{P}_{\mathrm{H}^2}$ is the projection onto the Hardy subspace $\mathrm{H}^2(\mathbb{T}, \mu)$ ([1], [11]). Then it is easy to see that $\delta(x) \in \mathscr{K}(\mathscr{H}) = \mathscr{J}(\mathscr{B}(\mathscr{H}))$ for $x \in \mathrm{C}(\mathbb{T})$ and that δ is not implemented by a compact operator.

We will, however, show in this paper that if N is self-adjoint and w-closed in M, then, except for certain situations, all derivations of N into $\mathscr{J}(M)$ are inner. Moreover, for the most typical excepted case we'll construct a counterexample.

This derivation problem was initiated in the case $M = \mathscr{B}(\mathscr{H})$ and $\mathscr{J}(M) = \mathscr{K}(\mathscr{H})$ by Johnson and Parrott in a paper of the early '70s ([3]). In that paper Johnson and Parrott wanted to characterize the commutant modulo the ideal of compact operators $\mathscr{K}(\mathscr{H}) \subseteq \mathscr{B}(\mathscr{H})$ for a von Neumann algebra $N \subseteq \mathscr{B}(\mathscr{H})$. They noted that in order to identify it with the compact perturbations of the commutant of N in $\mathscr{B}(\mathscr{H})$, it suffices to show that any derivation δ: $N \mapsto \mathscr{K}(\mathscr{H})$ is inner. They proved that this is indeed the case if N has no certain type II_1 factors as direct summands. To do this they first solved the case when N is abelian, the other cases being rather easy consequences of it. The general type II_1 case was proved recently in [7] by different techniques and using more of the ergodic theory of the type II_1 factors.

In [4] this derivation problem is studied in the more general setting when $\mathscr{B}(\mathscr{H})$ is replaced by a semifinite von Neumann algebra, $\mathscr{K}(\mathscr{H})$ by the ideal $\mathscr{J}(M)$, and the center of N is assumed to contain the center of M. Under this hypothesis it is proved that if N is either an abelian or a properly infinite von Neumann algebra, then any derivation of N into $\mathscr{J}(M)$ is inner.

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