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ON THE REGULARITY OF INVERSES OF SINGULAR INTEGRAL OPERATORS

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1. Introduction. On $\mathbb{R}^d$ consider a convolution operator $Sf = f * K$ where K is a distribution homogeneous of degree $-d$. Suppose that S extends to an operator bounded and invertible on $L^2(\mathbb{R}^d)$. The inverse of S is given by convolution with a distribution L, also homogeneous of degree $-d$. In this paper it will be shown that any smoothness possessed by K is shared by L. More precisely, let L^r_γ denote the Sobolev space of functions possessing γ derivatives in L^r. We will show that if $K \in L^r_\gamma(S^{d-1})$ for some $\gamma > 0$ and $r \in (1, \infty)$, then also $L \in L^r_\gamma(S^{d-1})$. Moreover, the corresponding result holds in the setting of graded nilpotent Lie groups; it is the nonabelian case with which we are primarily concerned.

In order to formulate our theorem precisely in the nilpotent setting, several definitions are required. Let g be a graded finite-dimensional nilpotent Lie algebra. By *graded* we mean that g admits a vector space direct sum decomposition $g = \bigoplus_{j \in \mathbb{Z}^+} g_j$ with the property that $[g_i, g_j] \subset g_{i+j}$ for all $i, j \in \mathbb{Z}^+$. Let G be the unique connected, simply connected nilpotent Lie group with Lie algebra g, and let $D = \Sigma j \cdot \dim(g_j)$ denote its homogeneous dimension and $d = \Sigma \dim(g_j)$ its dimension as a vector space. Identify g henceforth with the algebra of left-invariant vector fields on G, and in turn identify the left-invariant vector fields with G itself via the exponential map based at the group identity element, 0. Fix a basis $\{Y_{ji}: 1 \leqslant i \leqslant \dim(g_j)\}$ as a vector space, with each $Y_{ji} \in g_j$. This basis for g establishes a canonical coordinate system $x = (x_{ji})$ on G via the last two identifications, hence identifies G with $\mathbb{R}^d$. Haar measure equals Lebesgue measure in these coordinates, and all integration over G in this paper will be with respect to Haar measure. On G define dilations $\{\delta_r: r > 0\}$ by $\delta_r(x) = (r^j x_{ji})$; the map $r \mapsto \delta_r$ is a group homomorphism from $\mathbb{R}^+$ to the automorphism group of G. Define $\|x\| = r^{-1}$, where r is the unique positive number satisfying $\Sigma_{i,j} r^{2j} x_{ji}^2 = 1$ for all $x \neq 0$, and $\|0\| = 0$.

Let $\mathscr{S}$ and $\mathscr{S}'$ denote respectively the Schwartz space and the space of tempered distributions on G, defined via the identification with $\mathbb{R}^d$. Temporarily setting $f^r(x) = f(\delta_r x)$, we say that a distribution $K \in \mathscr{S}'$ is homogeneous of degree α if $\langle K, f^r \rangle = r^{-\alpha - D}\langle K, f \rangle$ for all $f \in \mathscr{S}$. Given $f \in \mathscr{S}$, set $f^{(x)}(y) = f(xy^{-1})$ and define $f * K(x) = \langle K, f^{(x)} \rangle$ for $K \in \mathscr{S}'$, so that when K is a function, $f * K(x) = \int f(xy^{-1})K(y)\,dy$. Now change notation and let $f^{(a)}(x) =$

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