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THE SMOOTH SURFACES IN $\mathbf{P}^4$ WITHOUT APPARENT TRIPLE POINTS

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§0. Introduction.

Definition 0.1. A surface in $\mathbf{P}^4$* is said to be without apparent triple points if the union of its trisecant lines does not fill $\mathbf{P}^4$.

In this paper we will show the following:

THEOREM 0.2. *The smooth surfaces in* $\mathbf{P}^4$ *without apparent triple points are the elliptic quintic scrolls and the surfaces contained in quadric hypersurfaces.*

Remark 0.3. It is well known (see [Ro, p. 151]) that a surface contained in a quadric hypersurface is either a complete intersection of the quadric and another hypersurface or linked to a plane by the quadric and another hypersurface.

This problem, without the assumption that the surfaces being considered are smooth, has been studied by Severi ([Se]). His result is that the trisecant lines of a surface in $\mathbf{P}^4$ without apparent triple points generate a hyperquadric or a hypersurface ruled in planes, each plane intersecting the surface in a curve of degree 3 or more. His approach was to characterize the hypersurfaces in $\mathbf{P}^4$ containing a 3-dimensional family of lines.

Z. Ran has shown ([Ra]) that a codimension-2 variety in $\mathbf{P}^N$ defined by a section of a rank-2 vector bundle is without apparent $(N-1)$-tuple points if and only if it is contained in a hypersurface of degree $N-2$. Moreover, when this is so, the variety is a complete intersection.

We will not use these results in the following. Our method will be to study the sectional genus of a surface: We show that a surface in $\mathbf{P}^4$ without apparent triple points and not of one of the types stated in Theorem 0.2 has to be birationally ruled over a curve of genus 2 or more. We show next that it is contained in a cubic hypersurface. Finally, by using a formula concerning quadrisecant lines, we show that such a surface without apparent triple points does not exist.

It is an open question whether there exists any smooth birationally ruled surface of genus 2 or more in $\mathbf{P}^4$.

This work is a part of [A]. Theorem 0.2 was conjectured by C. Peskine, and I am grateful for his advice and suggestions. Furthermore, I would like to thank

*Projective 4-space over an algebraically closed field of characteristic zero.

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