Vol. 57, No. 1 DUKE MATHEMATICAL JOURNAL © August 1988

ZETA FUNCTIONS OF KUGA FIBER VARIETIES

SALMAN ABDULALI

Dedicated to Professor Michio Kuga on his sixtieth birthday

A Kuga fiber variety is a family of abelian varieties $f\colon A \to V$ parametrized by an arithmetic variety $V = \Gamma \setminus X$ and constructed from a symplectic representation $\rho\colon G \to \mathrm{Sp}(F, \beta)$ of an algebraic group G [8]. The zeta functions of such abelian schemes have been investigated by Kuga, Shimura, Deligne, Langlands, and Ohta [5, 11, 12, 16, 17]. In this paper we find relations between the zeta functions of two Kuga fiber varieties over the same base.

We shall now describe our main results. Let $f_1\colon A_1 \to V$ and $f_2\colon A_2 \to V$ be Kuga fiber varieties satisfying the H_2-condition (1.2) and defined over an algebraic number field k_0. We show that for a point $P \in V$, the action of the geometric fundamental group $\pi_1(V^{an}, P)$ on $H^*(A_{i,P}, \mathbf{Q})$ essentially determines the action of the arithmetic fundamental group $\pi_1(V, P)$ on $H^*(A_{i,P}, \mathbf{Q}_\ell)$. More precisely, suppose $F_1 \subset H^{b_1}(A_{1,P}, \mathbf{Q})$ and $F_2 \subset H^{b_2}(A_{2,P}, \mathbf{Q})$ are isomorphic $\pi_1(V^{an}, P)$-submodules. Then there exists a finite extension k of k_0 such that $F_1 \otimes \mathbf{Q}_\ell$ and $F_2 \otimes \mathbf{Q}_\ell((b_2 - b_1)/2)$ are isomorphic $\pi_1(V_k, P)$-submodules of $H^{b_1}(A_{1,P}, \mathbf{Q}_\ell)$ and $H^{b_2}(A_{2,P}, \mathbf{Q}_\ell)((b_2 - b_1)/2)$, respectively. This leads to relations between the zeta functions of A_1 and A_2 which we describe in §3.2.

Our proof is based on the fact that any $\pi_1(V^{an}, P)$-invariant rational cycle in the cohomology of a fiber of a Kuga fiber variety satisfying the H_2-condition is an absolute Hodge cycle (Proposition 1.3 and [6], Main Theorem 2.11).

Now, suppose that V_1 and V_2 are quaternion Hilbert modular curves. Then there are Kuga fiber varieties $W_1 \to V_1$ and $W_2 \to V_2$ whose zeta functions are known. If V_1 and V_2 are suitably chosen, then there exists a Kuga fiber variety $A_2 \to V_1 \times V_2$ which is not a product [10]. We note that the results of [12] are not applicable in this situation, since $V_1 \times V_2$ is not simple. In §4 we compare the zeta functions of $A_1 = W_1 \times W_2$ and A_2 in some special cases.

The first two sections are mainly a review of known facts; we prove our main results in §3 and give examples in §4.

This paper contains results from my thesis [1], which was written under the guidance of Professor Michio Kuga; I am deeply grateful for his help and encouragement.

Notations and conventions. All algebraic varieties are assumed to be smooth and connected. If X is a variety over a subfield of $\mathbf{C}$, then X^{an} denotes the

Received March 13, 1987. Revision received July 6, 1987. Research partially supported by a Sloan Doctoral Dissertation Fellowship at the State University of New York at Stony Brook and by the N.S.F. at the University of Chicago.