A Kuga fiber variety is a family of abelian varieties \( f: A \to V \) parametrized by an arithmetic variety \( V = \Gamma \setminus X \) and constructed from a symplectic representation \( \rho: G \to \text{Sp}(F, \beta) \) of an algebraic group \( G \) [8]. The zeta functions of such abelian schemes have been investigated by Kuga, Shimura, Deligne, Langlands, and Ohta [5, 11, 12, 16, 17]. In this paper we find relations between the zeta functions of two Kuga fiber varieties over the same base.

We shall now describe our main results. Let \( f_1: A_1 \to V \) and \( f_2: A_2 \to V \) be Kuga fiber varieties satisfying the \( H_2 \)-condition (1.2) and defined over an algebraic number field \( k_0 \). We show that for a point \( P \in V \), the action of the geometric fundamental group \( \pi_1(V, P) \) on \( H^*(A_i, \mathbb{Q}) \) essentially determines the action of the arithmetic fundamental group \( \pi_1(V, P) \) on \( H^*(A_i, \mathbb{Q}) \). More precisely, suppose \( F_1 \subset H^b_1(A_1, \mathbb{Q}) \) and \( F_2 \subset H^b_1(A_2, \mathbb{Q}) \) are isomorphic \( \pi_1(V, P) \)-submodules. Then there exists a finite extension \( k \) of \( k_0 \) such that \( F_1 \otimes \mathbb{Q}_k \) and \( F_2 \otimes \mathbb{Q}_k ((b_2 - b_1)/2) \) are isomorphic \( \pi_1(V_k, P) \)-submodules of \( H^b_1(A_1, \mathbb{Q}_k) \) and \( H^b_1(A_2, \mathbb{Q}_k) ((b_2 - b_1)/2) \), respectively. This leads to relations between the zeta functions of \( A_1 \) and \( A_2 \) which we describe in §3.2.

Our proof is based on the fact that any \( \pi_1(V, P) \)-invariant rational cycle in the cohomology of a fiber of a Kuga fiber variety satisfying the \( H_2 \)-condition is an absolute Hodge cycle (Proposition 1.3 and [6], Main Theorem 2.11).

Now, suppose that \( V_1 \) and \( V_2 \) are quaternion Hilbert modular curves. Then there are Kuga fiber varieties \( W_1 \to V_1 \) and \( W_2 \to V_2 \) whose zeta functions are known. If \( V_1 \) and \( V_2 \) are suitably chosen, then there exists a Kuga fiber variety \( A_2 \to V_1 \times V_2 \) which is not a product [10]. We note that the results of [12] are not applicable in this situation, since \( V_1 \times V_2 \) is not simple. In §4 we compare the zeta functions of \( A_1 = W_1 \times W_2 \) and \( A_2 \) in some special cases.

The first two sections are mainly a review of known facts; we prove our main results in §3 and give examples in §4.

This paper contains results from my thesis [1], which was written under the guidance of Professor Michio Kuga; I am deeply grateful for his help and encouragement.

Notations and conventions. All algebraic varieties are assumed to be smooth and connected. If \( X \) is a variety over a subfield of \( \mathbb{C} \), then \( X^{an} \) denotes the