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POLYNOMIAL PROPER MAPS BETWEEN BALLS

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Introduction. The purpose of this paper is to investigate the structure of polynomial proper maps between balls in (perhaps) different dimensional complex vector spaces. There are two main results. First we verify a statement made by Forstnerič [F] that for each odd positive integer there is a monomial proper map that is invariant under a certain matrix group. Second, we verify that a general conjecture [D] about proper holomorphic maps holds in the context of polynomial proper maps.

The general conjecture about proper maps is that a proper holomorphic map between balls that is sufficiently regular at the boundary must admit a finite factorization that plays the role of a finite Blaschke product in one variable, and as a consequence must be rational. This factorization consists of a finite application of simple operations; these operations are composition with automorphisms, composition with certain linear maps between different dimensional spaces, and the extend map and its inverse. The extend map amounts to multiplication of a component by each of the components of an automorphism, and hence, unless we are in one variable, changes the target dimension. We show that every polynomial proper map admits a finite factorization of the type above, without requiring automorphisms that move the origin. We first prove the result for monomials and then determine which linear transformations can be used to reduce the polynomial case to the monomial case.

The other main result involves the invariance of monomial maps under finite unitary groups. Let G_k be the cyclic subgroup of $U(2)$, the unitary 2-by-2 matrices, generated by the diagonal matrix with eigenvalues $\exp(2\pi i/k)$ and $\exp(4\pi i/k)$. Then, for each odd integer $k = 2r + 1$, we exhibit an invariant holomorphic map f that maps the 2-ball properly to the $(2 + r)$-ball. When $r = 0$, we have the identity map. Forstnerič listed nontrivial maps for $r = 1, 2, 3$, and 4 and remarked that he believes there is such an example for each r. We prove this here and show that the coefficients are essentially uniquely determined. We give a combinatorial formula for (the squares of) these coefficients and an interesting asymmetric analog of Pascal's triangle. These maps could turn out to be particularly interesting for the following reason. When r is 0, 1, or 2, the corresponding proper map from B_2 to B_{2+r} is the monomial mapping of largest degree. If this were true in general, several nice corollaries would follow. We leave this question to a future paper.

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