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EQUIVARIANT ALGEBRAIC VS. TOPOLOGICAL K-HOMOLOGY ATIYAH–SEGAL-STYLE

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Consider a linear algebraic group G acting on a reasonable noetherian scheme X. The category of coherent algebraic G-modules on X gives rise to an equivariant algebraic K-theory $G(G, X)$. The main result of this paper is that if one reduces this mod ℓ^ν and inverts the action of the Bott element β to impose Bott periodicity, the resulting $G/\ell^\nu(G, X)[\beta^{-1}]$ is a sort of Atiyah–Segal-style equivariant topological K-homology with respect to the étale topology of X. In particular, it admits a Fary spectral sequence for the étale topology on the orbit topos X/G, and a Cartan–Leray spectral sequence for isovariant étale covers of X. For X regular and of finite type over the complex numbers $\mathbb{C}$, or for X equivariantly quasiprojective over $\mathbb{C}$, $G/\ell^\nu(G, X)[\beta^{-1}]$ is in fact equivalent to the equivariant generalized homology theory with locally compact supports dual to the equivariant K-cohomology with compact supports considered by Segal in [Seg 1]. This equivariant homology theory has also been considered recently by Ginsburg and by Kazhdan and Lusztig in their work on representations of p-adic groups [KL]. The étale version $G/\ell^\nu(G, X)[\beta^{-1}]$ of this homology theory has the usual advantages of working over more general base schemes than $\mathbb{C}$ and of admitting natural Galois actions. I verify that this theory satisfies the Kazhdan–Lusztig axioms in 3.20 below; this is the first essentially complete verification that any such theory exists.

This paper complements the results of [T5], where I showed the completion $G/\ell^\nu_*(G, X)[\beta^{-1}]^\wedge_{IG}$ is the equivariant topological K-homology of X in the sense of Borel. Homology equivariant in the sense of Borel is a coarser invariant than homology equivariant in the sense of Bredon as considered in this paper. The Atiyah–Segal completion theorem of [AS] says in the topological classical case that the Borel equivariant K-cohomology is the completion of the Bredon-equivariant K-cohomology with respect to the augmentation ideal IG of the representation ring of G. The corresponding result in algebraic geometry results from comparing this paper with [T5].

The key idea of the comparison between the algebraic and the topological comes from the main comparison theorem of [T1]. The main new idea in the equivariant comparison is the construction of the orbit topos "X/G" for G acting on X. The points of "X/G" correspond to the orbits of G in X, whether the orbits are closed or not. The usual existence of nonclosed orbits shows that

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