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DIOPHANTINE APPROXIMATION ON HYPERBOLIC ORBIFOLDS

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Several recent papers have looked at diophantine approximation, and especially the Markoff theory, from the point of view of the geometry of geodesics on hyperbolic surfaces. The basic idea is to relate the approximability of a real number to the excursions of a geodesic into the cusp region on an associated hyperbolic surface. Here we will continue the investigation begun in [H1] by considering geodesic excursions on hyperbolic orbifolds. In this context the problem only appears as a geometric one and does not relate back to classical diophantine approximation. Nonetheless, we will see that there exists a natural geometric analogue to the Markoff spectrum and we shall show that a Markoff type theorem holds for this spectrum.

In [Sh] M. Sheingorn introduced a new surface on which the Markoff theory can be realized geometrically, and he used this surface in a pivotal way to describe the relationship between the different surfaces that have been studied. One interesting result of his investigations is a theorem which says, in a sense we shall later make precise, that the surfaces involved have the same simple closed geodesics. Here we consider surfaces similar to those studied in [Sh] except that punctures may be replaced either by branch type singularities or by holes which cause the surfaces to have infinite area. Using geometric and topological methods that are independent of the classical Markoff theory, we show that these surfaces have the same simple closed geodesics. An immediate consequence is that the principal results of [H1] extend to Sheingorn's pivotal $(0; 2, 2, 2, \infty)$ surfaces. It then remains to consider the surfaces on which singularities have been introduced.

To a geodesic α on one of the surfaces we associate a value $D(\alpha)$ that describes the closest approach of the geodesic to a singularity on the surface. The set of values $D(\alpha)$ is called the Markoff–Cohn spectrum of the surface. The main result of this paper is that there is a Markoff type theorem for each of the surfaces. In other words, there is a number D_0 so that the set of values in the Markoff–Cohn spectrum greater than D_0 is discrete and has D_0 as a limit point. Moreover, we show that D_0 depends only on the topology of the surface and the order of the singularities, and thus remains constant under deformations of hyperbolic structure. As in [H1], the set of geodesics α with $D(\alpha) \geqslant D_0$ is the closure of the set of

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