Vol. 56, No. 3 DUKE MATHEMATICAL JOURNAL © June 1988

ON THE NUMBER OF PRIME FACTORS OF AN INTEGER

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1. Introduction. Let $\pi(x, k)$ denote the number of positive integers not exceeding x that have exactly k distinct prime factors. Estimates for $\pi(x, k)$ have been given by a number of authors. Landau [7, p. 211] showed that for fixed k the asymptotic formula

$$\pi(x, k) \sim \frac{x}{\log x} \cdot \frac{(\operatorname{loglog} x)^{k-1}}{(k-1)!} (x \to \infty)$$

holds. Sathe [10] and Selberg [11] proved a more precise quantitative estimate. Let

$$F(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \frac{z}{p-1}\right)\left(1 - \frac{1}{p}\right)^z, \tag{1.1}$$

where the product is taken over all primes p. Then

$$\pi(x, k) = F(y) \frac{x}{\log x} \cdot \frac{(\operatorname{loglog} x)^{k-1}}{(k-1)!} \left(1 + O\left(\frac{1}{\operatorname{loglog} x}\right)\right) \tag{1.2}$$

holds uniformly for $x \geqslant 3$ and $1 \leqslant k \leqslant C \operatorname{loglog} x$, for any given fixed $C > 0$, where here and in the sequel we set

$$y = \frac{k}{\operatorname{loglog} x}. \tag{1.3}$$

The Sathe–Selberg result remained for a long time the strongest of its kind. An extension beyond the range $k \leqslant C \operatorname{loglog} x$ has been obtained only quite recently by Hensley [4], who proved that

$$\pi(x, k) = F(y) \frac{x}{\log x} \cdot \frac{(\log_2 x)^{k-1}}{(k-1)!} \exp\left\{-\frac{1}{2} k \left(\frac{\log_3 x}{\log_2 x}\right)^2\right\} \left(1 + O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right) \tag{1.4}$$

Received August 11, 1986. Revision received January 5, 1987.