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SPECIAL K-TYPES, TEMPERED CHARACTERS AND THE BEILINSON–BERNSTEIN REALIZATION

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§1. Introduction. Let $\mathscr{G}$ be a connected linear semisimple Lie group (the precise assumptions on $\mathscr{G}$ will be described in §2) and $\mathscr{K}$ be a maximal compact subgroup of $\mathscr{G}$. As an invariant attached to representations of $\mathscr{G}$, Vogan introduced in [25] a notion of "lowest" $\mathscr{K}$-types and used it to give a classification of irreducible admissible representations for $\mathscr{G}$. In this theory, roughly speaking, an irreducible representation is specified in terms of the lowest $\mathscr{K}$-types and is realized as some subquotient of a certain induced representation. In contrast to Langlands–Knapp–Zuckerman's classification ([16], [17]), Vogan's theory is technically more algebraic. Using the language of $\mathscr{D}$-module, Beilinson and Bernstein introduced in [3] a geometric theory for general modules over $\mathfrak{g}$, the complexified Lie algebra of $\mathscr{G}$, in which one can "localize" a $\mathfrak{g}$-module to a certain sheaf of $\mathscr{D}$-module on the flag variety X associated to $\mathfrak{g}$. A $\mathfrak{g}$-module can thus be realized as the space of global sections of a certain $\mathscr{D}$-module on X. From the geometric point of view, this construction is most natural and most simple. In this paper, we study $\mathscr{K}$-types of an arbitrary induced standard Harish–Chandra module (i.e., $(\mathfrak{g}, \mathscr{K})$-module) via Beilinson–Bernstein's construction. Our goal is to search for certain "special" $\mathscr{K}$-types from the geometric point of view and to put Vogan's classification into the more geometric context.

To be more precise, we fix a complexification $K \subset G$ for the pair $\mathscr{K} \subset \mathscr{G}$. K acts on the flag variety X. In the Beilinson–Bernstein theory, a quasisimple Harish–Chandra module is localized to a $(\mathscr{D}_\lambda, K)$-module on X for some dominant linear form λ on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; here $\mathscr{D}_\lambda$ is the twisted sheaf of differential operators (t.d.o. for short) on X parametrized by λ. On the other hand, to each K-orbit Q and a λ-compatible connection τ on Q, there is an associated standard module $\mathscr{I}_{Q,\tau,\lambda}$ which contains a unique irreducible submodule denoted by $\mathscr{L}_{Q,\tau,\lambda}$. We have $\Gamma(\mathscr{L}_{Q,\tau,\lambda}) \subset \Gamma(\mathscr{I}_{Q,\tau,\lambda})$. The nontrivial $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$'s exhaust all the irreducible Harish–Chandra modules. With these, we can phrase our goal more precisely as: From a geometric point of view, find a certain set of "special" K-types in $\Gamma(\mathscr{I}_{Q,\tau,\lambda})$ which will "locate" the irreducible submodule $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$ (when nontrivial), and find explicit formulae for these K-types. Of course, these special K-types will be nothing but the lowest K-types in Vogan's sense.

Our results can be best explained by our methods. In the first part, we establish the following:

(a) A criterion for $\Gamma(\mathscr{L}_{Q,\tau,\lambda})$ to be nontrivial (Theorem 3.15; this is an unpublished result of Beilinson and Bernstein).

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