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SELBERG TRACE FORMULAE, PSEUDODIFFERENTIAL OPERATORS, AND GEODESIC PERIODS OF AUTOMORPHIC FORMS

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Let X_Γ be a compact hyperbolic surface: $X_\Gamma = \Gamma \backslash \mathfrak{h}$, with $\mathfrak{h}$ the upper $1/2$ plane and Γ a discrete co-compact subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Associated to X_Γ are two different kinds of objects:

(a) spectral data of its Laplacian Δ: numerically, the spectrum $\mathrm{spec}_\Delta(X)$ of eigenvalues $\{\lambda_k\}$; "geometrically," the normalized eigenfunctions $\{u_k\}$ of Δ;

(b) geodesic flow G^t: numerically, the period spectrum $\{L_\gamma\}$ of lengths of closed geodesics; "geometrically," the closed geodesics $\{\gamma\}$ and the ergodic properties of G^t.

Selberg's well-known trace formula provides a link between the numerical data of (a) and (b). To state the formula, let $R = \sqrt{-\left(\Delta + \frac{1}{4}\right)}$ and let $\mathrm{spec}(R) = \{r_k\}$. Then, for suitable f (including $f \in C_0^\infty(\mathbb{R})$), one has the

Selberg trace formula (STF):

$$\mathrm{Tr}\,\hat{f}(R) = \sum_k \hat{f}(r_k) = \frac{A}{4\pi}\int_{-\infty}^{\infty} \hat{f}(r) r \tanh \pi r \, dr + \sum_{\{\gamma\}} \frac{L_{\gamma_0} f(L_\gamma)}{\mathrm{sh}\, L_{\gamma/2}},$$

where γ_0 is the primitive closed geodesic corresponding to γ (once around) and A is the area.

By choosing f suitably, one can deduce precise asymptotic formulae for the spectral function $N(\lambda) = \sum_{\sqrt{\lambda_k} \leqslant \lambda} 1$ and the length spectral function $\mathscr{V}(T) = \sum_{L_\gamma \leqslant T} L_\gamma$. (See, e.g., [H] for these and many other applications of STF.)

Our purpose in the present paper is to generalize the trace formula so as to provide a link between the geometric data of (a) and (b). By *geometric* we mean the following: A closed geodesic γ, as a curve in the unit tangent bundle $S^*(X_\Gamma) = \Gamma \backslash \mathrm{PSL}_2(\mathbb{R})$, defines a probability measure $d\mu_\gamma$ on $C(S^*X_\Gamma)$: $\int a\, d\mu_\gamma = (L_\gamma)^{-1}\int_\gamma a$. Less obviously, an eigenfunction u_k of Δ defines a "pseudomeasure" dU_k on $C(S^*X_\Gamma)$ (cf. [Z1]). The $\{dU_k\}$ are defined via pseudodifferential operator (ψDO) theory. A calculus of ψDO's on $L^2(X_\Gamma)$ is an assignment $a \to \mathrm{Op}(a)$ of bounded operators to symbols $a \in C^\infty(S^*X_\Gamma)$, thought of as homogeneous functions of order 0 in $C^\infty(T^*X_\Gamma)$ (for purposes of trace formulae

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