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DEGENERATIONS OF THE HYPERBOLIC SPACE

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1. Introduction. Let $\mathbb{H}^n$ denote the n-dimensional hyperbolic space, i.e., the only complete, simply connected Riemannian manifold with constant sectional curvature -1. A model (the Poincaré model) for $\mathbb{H}^n$ is the open unit ball

$$\{(x_1, x_2, \ldots, x_n) \in R^n;\ x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}$$

in $\mathbb{R}^n$ with Riemannian metric given by $ds = dx/(1 - r^2)$ at a point at distance r from the origin. We never use this formula in the paper.

Denote by $\mathrm{Isom}_+\mathbb{H}^n$ the Lie group of orientation-preserving isometries of $\mathbb{H}^n$ (it is isomorphic to $SO_0^+(n,1)$). Let G be a (discrete) group. Any discrete and faithful representation α: $G \to \mathrm{Isom}_+\mathbb{H}^n$ gives rise to a hyperbolic manifold $\mathbb{H}^n/\alpha(G)$ and an isomorphism $G \approx \pi_1(\mathbb{H}^n/\alpha(G))$ that is well defined up to conjugation (the isomorphism depends on the choice of basepoint). If β: $G \to \mathrm{Isom}_+\mathbb{H}^n$ is a conjugate of α, i.e., for some $h \in \mathrm{Isom}_+\mathbb{H}^n$ $\beta(g) = h^{-1}\alpha(g)h$ ($g \in G$), then the hyperbolic manifolds $\mathbb{H}^n/\alpha(G)$ and $\mathbb{H}^n/\beta(G)$ are isometric, and the isometry induces an isomorphism on fundamental groups that commutes (up to conjugation) with the preferred isomorphisms $G \approx \pi_1(\mathbb{H}^n/\alpha(G))$ and $G \approx \pi_1(\mathbb{H}^n/\beta(G))$. Therefore, the set $\mathscr{H}^n(G)$ of "homotopy n-hyperbolic structures" on G can be defined as

(1) the set of conjugacy classes of discrete and faithful representations $G \to \mathrm{Isom}_+\mathbb{H}^n$; or

(2) the set of equivalence classes of pairs (M, ϕ), where M is a hyperbolic n-manifold and ϕ is an isomorphism between G and $\pi_1(M)$ defined up to conjugation (pairs (M, ϕ) and (N, Ψ) are equivalent if there is an isometry f: $M \to N$ such that the diagram

$$\begin{array}{ccc} & & \pi_1(M) \\ & \nearrow & \\ G & & \downarrow f_\# \\ & \searrow & \\ & & \pi_1(N) \end{array}$$

commutes up to conjugation); or

(3) the set of equivalence classes of pairs (M, ϕ), where M is a hyperbolic n-manifold and ϕ is a homotopy equivalence $K(G,1) \to M$ (pairs (M, ϕ) and

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