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ISOMORPHISMS MODULO THE COMPACT OPERATORS OF NEST ALGEBRAS II

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Consider three operator algebras associated with a nest $\mathcal{N}$: $\mathcal{T}(\mathcal{N}) = \operatorname{Alg}\mathcal{N}$, $\mathcal{QT}(\mathcal{N}) = \mathcal{T}(\mathcal{N}) + \mathcal{K}$, and $\mathcal{A}(\mathcal{N}) = \mathcal{QT}(\mathcal{N})/\mathcal{K}$. In this paper, isomorphisms of $\mathcal{A}(\mathcal{N})$ onto another such algebra $\mathcal{A}(\mathcal{M})$ are shown to be of the form $\operatorname{Ad} u \circ \alpha$, where u is a unitary element of the Calkin algebra and α is an automorphism of $\mathcal{A}(\mathcal{N})$ that preserves the nest in a way to be described precisely later on.

The corresponding problem for the other two algebras has already been solved. In his seminal paper on nest algebras, Ringrose [26] showed that rank-one elements of $\mathcal{T}(\mathcal{N})$ are characterized in an algebraic way intrinsic to $\mathcal{T}(\mathcal{N})$. This, together with the fact that there is an abundance of rank-one operators in $\mathcal{T}(\mathcal{N})$, allowed him to prove that every isomorphism α of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{T}(\mathcal{M})$ is of the form $\alpha(T) = STS^{-1}$. In particular, the operator S that takes the nest $\mathcal{N}$ onto $\mathcal{M}$ induces an order isomorphism θ_S of $\mathcal{N}$ onto $\mathcal{M}$ that preserves dimension. In [11] the second author established that every order isomorphism of $\mathcal{N}$ onto $\mathcal{M}$ that preserves dimension is implemented by a similarity. If isomorphisms α and β give rise to the same order isomorphism θ, then $\beta^{-1}\alpha$ is an automorphism of $\mathcal{T}(\mathcal{N})$ that preserves the nest. Such automorphisms are inner (of the form $\operatorname{Ad} S$ for S in $\mathcal{T}(\mathcal{N})^{-1}$). Thus, one obtains a characterization of the isomorphisms of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{T}(\mathcal{M})$ modulo the inner automorphisms of $\mathcal{T}(\mathcal{N})$ as the set of dimension-preserving order isomorphisms of $\mathcal{N}$ onto $\mathcal{M}$ [13].

For quasitriangular algebras, part of the problem is easy. If α is an isomorphism of $\mathcal{QT}(\mathcal{N})$ onto $\mathcal{QT}(\mathcal{M})$, then it takes the unique minimal ideal, the finite-rank operators, onto itself. By a theorem of Rickart [25, 2.5.19], this map is implemented by a similarity. However, to make any more progress requires more work because $\mathcal{N}$ cannot be recovered exactly from $\mathcal{QT}(\mathcal{N})$. For example, finitely many atoms of $\mathcal{N}$ of finite rank may be shuffled around without changing $\mathcal{QT}(\mathcal{N})$. A theorem of Andersen [1] was extended in [11, Theorem 2.2] to characterize when $\mathcal{QT}(\mathcal{N}) = \mathcal{QT}(\mathcal{M})$. Earlier, partial results had been obtained in [23]. Using this result, the second author and B. Wagner [13] characterized the automorphisms of $\mathcal{QT}(\mathcal{N})$ modulo inner automorphisms as a certain group of "almost isomorphisms" of $\mathcal{N}$.

The algebra $\mathcal{A}(\mathcal{N})$ has the same disadvantage that $\mathcal{QT}(\mathcal{N})$ has, that $\mathcal{N}$ cannot be exactly recovered. It further suffers grievously from a lack of finite-rank

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