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COMPLETE MINIMAL SURFACES WITH LONG LINE BOUNDARIES

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In this paper we will study complete minimal surfaces (CMS's) bounded by lines in $\mathbb{R}^3$. By complete we mean that a path that leaves every compact subset of a manifold has infinite length, even if the manifold has boundary. We will prove Bernstein-type theorems for such surfaces. For example, we prove that if M is a CMS whose interior is a graph over a square J in the (x, y) plane, and if the boundary of M is composed of the four vertical lines over the vertices of J, then M is Scherk's surface. Another theorem of this type that we prove is: Let M be a CMS whose interior is a graph over an infinite strip J in the (x, y) plane, and suppose ∂M is composed of two vertical lines over points of ∂J. Then M is part of a helicoid. We obtain other results of this nature.

Minimal surfaces bounded by lines have been studied extensively. A line L on a minimal surface in $\mathbb{R}^3$ has an intrinsic meaning: M is invariant by the symmetry of $\mathbb{R}^3$ through L (the reflection principle), hence L lifts to a geodesic in the universal conformal covering space of M, with the Poincaré metric.

In his memoir [5] Riemann found a means to construct minimal surfaces with boundary a given polygon, where the sides of the polygon could be short or long (i.e., of finite or infinite length). His technique works for polygons with up to four sides. The contemporary reader may interpret Riemann's statements of theorems as announcing unicity of CMS's with given polygonal boundaries (see Darboux [2], pp. 491–492); however, the paper of Riemann only addresses the problem of existence.

Serret found an infinite family of CMS's with boundary two lines, each example distinct from a helicoid and simply connected [7]. Riemann found such examples that are not simply connected, modelled on a punctured torus [5]. Jenkins and Serrin consider the Plateau problem over convex compact domains D in the plane where the data on ∂D is discontinuous [3]. More precisely, suppose ∂D is a polygon $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ (in the order indicated) and one desires a minimal surface M that is a graph over int D and takes the values $+\infty$ on each A_i, $-\infty$ on each B_i (this implies ∂M is the set of lines over the vertices of D). They prove that M exists and is unique provided $\Sigma_i^n |A_i| = \Sigma_i^n |B_i|$, where $|A|$ means the length of A. We will prove $c(M)$ is finite.

It would be interesting to know whether their techniques work on noncompact domains D.

I. The helicoid. The helicoid M is a CMS modelled on $\mathbb{C}$ having a Weierstrass representation $g(z) = e^z$, $\omega = -iae^{-z}\, dz$, a real. M is invariant by a vertical

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