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HAMILTONIAN STRUCTURE FOR THE MODULATION EQUATIONS OF A SINE-GORDON WAVETRAIN

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I. Introduction. We study the sine-Gordon equation,

$$\varepsilon^2(\partial_t^2 - \partial_x^2)u + \sin u = 0, \tag{I.1}$$

where ε is a small parameter.[1] This equation is an example of a conservative, nonlinear, dispersive wave equation which enjoys the additional special property that it is integrable as an (infinite-dimensional) Hamiltonian system. It has exact solutions in the form

$$u(x,t) = W_N\left(\frac{\vec{\theta}}{\varepsilon}(x,t); \vec{\kappa}, \vec{\omega}\right). \tag{I.2}$$

These solutions (I.2) depend upon $2N$ real parameters $\vec{\kappa} = (\kappa_1, \dots, \kappa_N)$ and $\vec{\omega} = (\omega_1, \dots, \omega_N)$ and N "phases,"

$$\theta_i(x,t) = \kappa_i x + \omega_i t + \theta_i^{(0)}.$$

The x and t dependence enters the waveform (I.2) only *linearly* through these phases. For each $\vec{\kappa}$ and $\vec{\omega}$, W_N is a real function on the N-torus T^N (2π-periodic in each θ_i/ε), which has an explicit representation in terms of the Riemann theta function. Because of this 2π periodicity, the parameters $\vec{\kappa}$ and $\vec{\omega}$ are interpreted as spatial wave numbers and temporal frequencies.

Thus, (I.2) represents a (real) $3N$-dimensional family of exact solutions of the sine-Gordon equation, each member of which is quasiperiodic in both space and time. We call this family of solutions the "*N-phase, quasiperiodic waves*." When $N = 1$, this family reduces to the well-known "periodic traveling waves" for the sine-Gordon equation. Because ε is small, these traveling waves are rapidly

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[1] For small ε, this problem is equivalent to the sine-Gordon equation $U_{TT} - U_{XX} + \sin U = 0$ on asymptotically long spatial and temporal scales, $X = x/\varepsilon$ and $T = t/\varepsilon$. We prefer the scaling (I.1) for fixed x and t as $\varepsilon \to 0$.