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UNIFORM DISTRIBUTION OF EIGENFUNCTIONS ON COMPACT HYPERBOLIC SURFACES

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0. Introduction. In this paper we will prove that the eigenfunctions $\{\varphi_k\}$ of the Laplacian on a compact hyperbolic surface X become uniformly distributed on X as $k \to \infty$, aside from a subsequence of "density" zero. To state the result precisely and to put the problem into its natural framework, we must begin by recalling some well-known analysis on compact Riemannian manifolds.

Let (X, g) be an n-dimensional compact Riemannian manifold, and let Δ be its Laplacian. Then $L^2(X) = \oplus_{\lambda_k} \mathscr{H}(\lambda_k)$, where $\Delta = -\lambda_k^2$ on $\mathscr{H}(\lambda_k)$ and $\dim \mathscr{H}(\lambda_k) = m(\lambda_k) < \infty$. Let $N(\lambda) = \Sigma_{\lambda_k \leqslant \lambda} m(\lambda_k)$. Then $N(\lambda) \sim (\omega_n/(2\pi)^n)(\mathrm{vol}(X))\lambda^n + R(\lambda)$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and where

(i) $R(\lambda) = O(\lambda^{n-1})$ in general;

(ii) $R(\lambda) = o(\lambda^{n-1})$ if the periodic geodesics of X form a set of measure 0 [D–G];

(iii) $R(\lambda) = O(\lambda^{n-1}/\log \lambda)$ if X is negatively curved [Bé].

The multiplicities $m(\lambda_k)$ are unknown except in special cases, and the most one can say in general is that $m(\lambda_k) \ll R(\lambda_k)$.

Let us next fix ordered orthonormal bases $\{\varphi_{k_i}: 1 \leqslant i \leqslant m(\lambda_k)\}$ for $\mathscr{H}(\lambda_k)$. To the resulting sequence $\{\varphi_{k_i}: k = 1, 2, 3, \ldots; 1 \leqslant i \leqslant m(\lambda_k)\}$ of orthonormal eigenfunctions we may associate a sequence of distributions $\{dU_{k_i}\}$ in $\mathscr{D}'(S^*X)$. This is done by means of pseudodifferential operator (ψDO) theory. Thus, we assume as given a calculus of ψDO's on X, i.e., an assignment Op: $C^\infty(S^*X) \to \mathscr{B}(L^2(X))$ of bounded operators $\mathrm{Op}(a)$ to smooth zeroth order symbols a, satisfying the usual requirements [Hö]. (A particularly natural calculus may be defined for hyperbolic surfaces, and we will be using that calculus exclusively in this paper (cf. [Z1])). By means of Op we can associate to a given eigenfunction φ_k the distribution dU_k defined by $\int_{S^*X} a\, dU_k = (\mathrm{Op}(a)\varphi_k | \varphi_k)$. We may then state a natural problem in the geometric asymptotics of Δ on X:

problem 1. What are the weak* limit points of the $\{dU_{k_i}\}$ (i.e., the $d\mu \in \mathscr{D}'(S^*X)$ for which there is a subsequence $\mathscr{S} \subset \{\lambda_{k_i}\}$ with $\lim_{\mathscr{S}} \int a\, dU_{k_i} = \int a\, d\mu$ for all a)?

It is well known that all such limit distributions are in fact invariant measures for the geodesic flow G^t on S^*X (cf. [Wi]). However, it is by and large unknown

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