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ON RATIONALLY DETERMINED LINE BUNDLES ON A FAMILY OF PROJECTIVE CURVES WITH GENERAL MODULI

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1. Let g, r, d be positive integers and let $\rho = \rho(g, r, d) = g - (r+1)(g+r-d)$. It is well known that if and only if either $\rho \geqslant 0$ and $r \geqslant 3$ or $g = 3$, $r = 2$, and $d = 4$, there is a unique irreducible component $\mathscr{H}$ of the Hilbert scheme of curves of degree d and arithmetic genus g in $\mathbb{P}^r$ such that

(i) there exists a nonempty, maximal Zariski open subset $\mathscr{H}_0$ of $\mathscr{H}$ such that all closed points of $\mathscr{H}_0$ are smooth points for the Hilbert scheme and correspond to smooth, irreducible, nondegenerate curves in $\mathbb{P}^r$;

(ii) the natural map $\mathscr{H}_0 \to \mathscr{M}_g$ of $\mathscr{H}_0$ into the moduli space of curves of genus g is dominant.

$\mathscr{H}$ is said to be a *component with general moduli* of the Hilbert scheme.

If $\pi\colon \mathscr{F} \to \mathscr{H}_0$ is the universal family over $\mathscr{H}_0$, both $\mathscr{F}$ and $\mathscr{H}_0$ are smooth schemes over the base field, which we shall assume to be the complex field; furthermore, the morphism π is smooth also. Let U be any nonempty Zariski open subset of $\mathscr{H}_0$ and let $\pi_U\colon \mathscr{F}_U \to U$ be the restriction of the universal family over U. The cokernel of the group morphism $\pi_U^*\colon \operatorname{Pic}(U) \to \operatorname{Pic}(\mathscr{F}_U)$ will be denoted by $\mathscr{R}(\mathscr{F}_U)$ and called *the group of rationally determined line bundles* on the curves of the family $\pi_U\colon \mathscr{F}_U \to U$. We notice that on $\mathscr{F}$ one has two naturally defined line bundles, namely ω, the relative canonical bundle of $\pi\colon \mathscr{F} \to \mathscr{H}_0$, and h, the hyperplane bundle corresponding to the morphism $\mathscr{F} \hookrightarrow \mathbb{P}^r \times \mathscr{H}_0$. We shall again denote by ω and h, if no confusion arises, the images of these line bundles in $\mathscr{R}(\mathscr{F})$ and in $\mathscr{R}(\mathscr{F}_U)$ for any open subset $U \subset \mathscr{H}_0$ via the natural restriction morphism $r_U\colon \mathscr{R}(\mathscr{F}) \to \mathscr{R}(\mathscr{F}_U)$.

The purpose of this paper is to prove

THEOREM (1.1). *Let $r \geqslant 3$, $g \geqslant 3$, and $\rho \geqslant 2$. Then for any nonempty Zariski open subset $U \subseteq \mathscr{H}_0$, $\mathscr{R}(\mathscr{F}_U)$ is generated by ω and h.*

What we shall actually prove is a slightly different assertion, equivalent to Theorem (1.1), which we are now going to state: Let $\mathscr{L}$ be any element in $\operatorname{Pic}(\mathscr{F})$, and let γ be a closed point in $\mathscr{H}_0$ corresponding to a smooth, complete curve Γ of degree d and genus g in $\mathbb{P}^r$, the fibre of π over γ. It is clear that $\mathscr{L}_\Gamma$, the restriction of $\mathscr{L}$ to Γ, only depends on the image of $\mathscr{L}$ in $\mathscr{R}(\mathscr{F})$. Conversely, we have

LEMMA (1.2). *Let $\mathscr{L}, \mathscr{L}' \in \operatorname{Pic}(\mathscr{F})$. If for every closed point $\gamma \in \mathscr{H}_0$ one has $\mathscr{L}_\Gamma \simeq \mathscr{L}'_\Gamma$, then the images of $\mathscr{L}$ and $\mathscr{L}'$ in $\mathscr{R}(\mathscr{F})$ coincide.*

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