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LOWER BOUNDS ON THE ENERGY OF MAPS

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Introduction. The first goal of this paper is to prove that for the standard metric on $\mathbb{RP}^n$ the identity map is energy-minimizing in its homotopy class. This is false for S^n as long as n is greater than 2. In fact, a recent result of Brian White shows that in dimensions greater than 2, if π_1 and π_2 are trivial, then in the homotopy class of the identity there are maps with arbitrarily small energy (see [W]). We prove our result by reformulating the energy functional $E(f)$ as an integral of the energy of geodesic segments over the space of geodesic segments of a fixed length (see Proposition 1).

For compact surfaces the identity map always minimizes energy in its homotopy class, since in two dimensions the energy is (sharply) bounded below by the area of the image. The other cases where the identity map is known to minimize energy are for compact Kähler manifolds and compact manifolds of nonpositive curvature (see [S]). The fact that the identity map on $\mathbb{RP}^n$ is a stable critical point for the energy functional was known. In fact, all space forms except for the round sphere have this property, as was shown by H. Urakawa in [U]. It was shown by Smith [S] that the index of Id for any Riemannian manifold is bounded below by $\mathrm{DIM}(C/\mathscr{k})$, where C represents the space of conformal vector fields (infinitesimal conformal transformations) and $\mathscr{k}$ the Killing fields. Hence, by conformal changes of the standard metric, we can find metrics on $\mathbb{RP}^n$ arbitrarily close to the standard metric where the identity map is not even a stable critical point.

It is easy to see by the result in this paper for $\mathbb{RP}^n$ that the identity map is energy-minimizing for those space forms covered by $\mathbb{RP}^n$. However, it is still unknown whether the identity map is energy-minimizing for those space forms of positive curvature that are not the sphere and not covered by $\mathbb{RP}^n$.

The argument in fact yields a more general sharp lower bound on the energy of any map from the standard $\mathbb{RP}^n$:

THEOREM 1. *Let f be a smooth map from $\mathbb{RP}^n$ with the standard metric to an arbitrary Riemannian manifold N. Then*

$$E(f) \geqslant \frac{1}{2} \cdot \left(\frac{L}{\pi}\right)^2 \cdot \mathrm{Vol}(\mathbb{RP}^n) = \left(\frac{L}{\pi}\right)^2 \cdot E(\mathrm{Id}),$$

where Id *is the identity map on $\mathbb{RP}^n$ and L is the length of the shortest closed*

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