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# A NOTE ON HYPERSPHERICAL MANIFOLDS OF POSITIVE CURVATURE AND GEOMETRIC CONTRACTIBILITY

MIKHAIL KATZ

**1.** Several different notions of "largeness" of a Riemannian manifold arise in the course of investigations, ranging from filling inequalities [$G_1$] to positive scalar curvature [GL]. Tying these different notions together is M. Gromov's "Vague Conjecture" on the volume of metric balls in a large manifold [$G_2$]. Meanwhile, it is interesting to explore the interrelationships that may exist among these various notions.

A complete Riemannian manifold $M^n$ is called *hyperspherical* (HS) if there exists a proper distance-decreasing map $f\colon M \to S^n_R$ with $\deg f \neq 0$ for any $R > 0$ (here a map is called *proper* if it maps a neighborhood of infinity in $M$ to a point of the sphere) [$G_2$, GL]. For example, the Euclidean plane $R^2$ is HS. The cylinder $S^1 \times R^1$ is not HS, but if you walk along it attaching a sequence of larger and larger bubbles as you go off to infinity, then the resulting manifold is HS. $M$ is called *geometrically contractible* (GC) if for every $\delta > 0$ there exists a number $\rho(\delta) > 0$ such that every ball $B(x, \delta) \subset M$ of radius $\delta$ can be contracted to a point in $B(x, \rho(\delta))$ [$G_1$, $G_2$]. For example, $R^2$ with the standard metric is GC (for $\rho(\delta) = \delta$), but $R^2$ with a complete metric of finite area is not. The cylinder with bubbles attached is not GC for the trivial reason that it is not simply connected, but even its universal cover is not GC: take a point $x$ at the base of a bubble and choose a ball $B(x, \delta)$ containing the thin neck. To retract this ball to a point, we must go all the way around the bubble.

An open question in [$G_2$] suggests that it is likely that HS manifolds with positive sectional curvature are GC. In section 2 we verify this suggestion for surfaces using geodesic loops and Gauss–Bonnet. In section 3 we describe a convex hypersurface of $R^4$ that is HS but not GC. It is still possible that $n$-dimensional HS manifolds of positive sectional curvature are $\mathrm{GC}_{n-2}$ [$G_2$].

**2.** Let $M$ be an open surface of positive curvature. Suppose $M$ is not GC. Then there is a number $\delta > 0$ and a sequence $x_j$ of points on $M$ such that the inclusion of $B(x_j, \delta)$ in $B(x_j, j\delta)$ is not homotopic to a constant map. It follows that the sequence $x_j$ goes to infinity and there is a closed curve $c_j \subset B(x_j, \delta)$ that is not freely homotopic to 0 in $B(x_j, j\delta)$. The curve $c_j$ is freely homotopic to the product of simple closed curves of the form $x_j a \cup ab \cup (x_j b)^{-1}$, where $a$ and $b$ are nearby points of $c_j$ and $x_j a$, $ab$, $x_j b$ are minimal arcs. Hence, given $\varepsilon > 0$, we may assume that each of these curves has length $< 2\delta + \varepsilon$. One of them, say

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