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ABELIAN VARIETIES WITH SEVERAL PRINCIPAL POLARIZATIONS

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0. Introduction. Let J denote the Jacobian variety of a smooth projective curve C and let θ denote the principal polarization associated to the Theta divisor of C in J. Then Torelli's theorem says that the pair (J, θ) determines the curve C up to isomorphism. A natural question would be: does J alone determine the curve C? This is not true in general. However there seem to be not many counterexamples in the literature, all of them in genus 2. The first examples are due to Humbert (cf. [5]), who studied abelian surfaces with real multiplication. Others are due to Hayashida and Nishi (cf. [3], [4]) who studied products of elliptic curves. In both cases the fact that a principally polarized abelian surface is either a Jacobian or a product of elliptic curves is heavily used.

In arbitrary dimensions there is only the general theorem of Narasimhan and Nori (cf. [10]) stating that any abelian variety admits only a finite number of principal polarizations up to isomorphism. There remains the problem: What is the actual number if isomorphism classes of principal polarizations of a given abelian variety? It is the aim of this paper to give a translation of this question into a number theoretical one. This gives a method for computing this number in many cases of which we will give several examples.

To state the results, let A be an abelian variety over the field of complex numbers. Let $\Pi(A)$ denote the set of isomorphism classes of principal polarizations of A and $\pi(A)$ the number of elements of $\Pi(A)$. In section 1 we show (cf. Theorem 1.5) that if A admits a principal polarization L_0, then L_0 induces a bijection between $\Pi(A)$ and the set of equivalence classes of totally positive symmetric (with respect to L_0) automorphisms of A modulo the natural action of $\mathrm{Aut}(A)$.

In section 2 we give a criterion (cf. Lemma 2.1 and Remark 2.2) for an abelian variety with real multiplication to admit a principal polarization, which easily can be applied to give examples for such varieties.

Hence Theorem 1.5 may be applied to give examples in the real multiplication case. In section 3 the number $\pi(A)$ is computed a little more in this case. It is shown that $\pi(A)$ is closely related to the class number h of the corresponding totally real field k. To be more precise, if $\mathrm{End}(A)$ equals the principal order in K, then $\pi(A) = h^+/h$, where h^+ denotes the narrow class number on K (cf. Theorem 3.1). As a corollary from this and Dirichlet's theorem we get $\pi(A) \leqslant$

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