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ON THE CUSPIDAL COHOMOLOGY OF ARITHMETIC SUBGROUPS OF SL(2n) AND THE FIRST BETTI NUMBER OF ARITHMETIC 3-MANIFOLDS

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1. Introduction. In the last decade, starting with the work of Harder, much attention has been devoted to the cohomology of arithmetic subgroups of SL(2, k) when k is a number field with nice arithmetic properties—for example, a totally real or a quadratic imaginary field. (Notable results are due in particular to Harder, Grunewald, Schwermer, Rohlfs, and most recently to Labesse–Schwermer [12], to whom we refer for bibliographical information).

In particular, in many instances, these authors prove the existence of *cuspidal* cohomology (a concept due to Eichler and Shimura) when the arithmetic subgroup is deep enough.

The purpose of this note is to prove the existence of cuspidal cohomology for deep arithmetic subgroups of SL(2, k), when k is *any* number field.

To describe our result, we use the following notation. If $\mathfrak{a}$ is a complex vector space, we denote by $\Lambda^{\bullet}\mathfrak{a} = (\Lambda^i\mathfrak{a})_{i\in\mathbf{Z}}$ the graded vector space such that $\Lambda^i\mathfrak{a}$ is the i-th exterior product of $\mathfrak{a}$ ($0 \leqslant i \leqslant \dim \mathfrak{a}$) and is equal for 0 for other values of i. (If $\mathfrak{a} = \{0\}$, it is natural to set $\Lambda^*\mathfrak{a} = \mathbf{C}$ in degree 0, 0 elsewhere). We write $\Lambda^{\bullet-m}\mathfrak{a}$ for the graded vector space defined by $(\Lambda^{\bullet-m}\mathfrak{a})_i = \Lambda^{i-m}\mathfrak{a}$. Finally, we use the notation $X^{\bullet}\otimes Y^{\bullet}$ for the usual tensor product of graded vector spaces.

Assume now that k is a number field with r_1 real places and r_2 complex places.

THEOREM 1. (i) *Assume Γ is a congruence subgroup of* SL(2, k). *Then the cuspidal cohomology $H^{\bullet}_{\text{cusp}}(\Gamma, \mathbf{C})$ is a finite sum of terms of the form*

$$\underbrace{\Lambda^{\bullet-1}0 \otimes \cdots \otimes \Lambda^{\bullet-1}0}_{r_1 \text{ factors}} \otimes \underbrace{\Lambda^{\bullet-1}\mathbf{C} \otimes \cdots \otimes \Lambda^{\bullet-1}\mathbf{C}}_{r_2 \text{ factors}}.$$

(ii) *If Γ is given, there is a congruence subgroup Γ' of Γ such that this sum is nonzero.*

In particular, for Γ deep enough, $H^q_{\text{cusp}}(\Gamma, \mathbf{C})$ is nontrivial in the range $r_1 + r_2 \leqslant q \leqslant r_1 + 2r_2$.

Of course part (i) is well known. It decomposes $H^{\bullet}_{\text{cusp}}$ into summands of the form

$$\bigotimes^{r_1}(\mathbf{C} \text{ in degree } 1) \otimes \bigotimes^{r_2}(\mathbf{C} \text{ in degrees } 1, 2).$$

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