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THE KÜNNETH THEOREM AND THE UNIVERSAL COEFFICIENT THEOREM FOR KASPAROV'S GENERALIZED K-FUNCTOR

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1. Introduction. Given C^*-algebras A and B with modest hypotheses, G. G. Kasparov [21] has defined groups $KK_j(A, B)$ ($j = 0, 1$) which play a fundamental role in the modern theory of C^*-algebras and, particularly, its application in areas related to global analysis and algebraic topology. In this paper we prove a Künneth Theorem which determines the Kasparov groups in terms of the periodic K-theory groups $K_*(B)$ of Karoubi and the Brown–Douglas–Fillmore groups $K^*(A)$, and we establish a Universal Coefficient Theorem (UCT) of the form

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(K_*(A), K_*(B)) \to KK_*(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

which determines the Kasparov groups in terms of K-theory. These short exact sequences are split, unnaturally. When $B = \mathbf{C}$ (the complex numbers) we obtain [with a new, perhaps simpler, proof] the UCT of L. G. Brown [5] for the Brown–Douglas–Fillmore groups $K^*(A) \cong KK_*(A, \mathbf{C})$.

A great deal of the power of Kasparov's theory comes from the existence of a "Kasparov intersection product" with good functorial properties, generalizing all of the usual products (cup, cap, slant, etc.) in topological K-theory. For reasons to be explained later, our UCT also determines this product structure. In particular, we determine the structure of the graded ring $KK_*(A, A)$.

Our results should be useful in several situations where the KK-groups are encountered. These include the classification of extensions of C^*-algebras, index theory on foliated manifolds (as in [10]), and index theory for elliptic operators with "coefficients" in a C^*-algebra (as in [25] and [30], §3B). For instance, a family of elliptic pseudodifferential operators over a compact manifold M with parameter space Y defines an element of $KK_*(C(M), C_0(Y))$; hence the computation of this group is of some interest. In Section 8 we discuss some applications to the "algebraic topology" of C^*-algebras. The computation of the graded ring $KK_*(A, A)$ and its graded module $KK_*(A \otimes A, A)$ in a few basic cases makes it possible for us to determine all of the "homology operations" and "admissible multiplications" for mod p K-theory of C^*-algebras.

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