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PERRON'S METHOD FOR HAMILTON-JACOBI EQUATIONS

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§1. Introduction. In this paper we are concerned with the existence of solutions of nonlinear first order scalar PDE's (Hamilton-Jacobi equations)

$$F(x, u, Du) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^N$ (we will actually replace $\mathbb{R}^N$ by a Banach space in Sections 2 and 3), F: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous, u: $\Omega \to \mathbb{R}$ is the unknown and Du denotes the gradient of u.

Proofs of the existence of global weak solutions of (1.1) in the literature rely on one of the following two methods. The most standard one is the vanishing viscosity method in which the desired solution of (1.1) is obtained as the limit of solutions u^ε of

$$-\varepsilon\Delta u^\varepsilon + F(x, u^\varepsilon, Du^\varepsilon) = 0 \quad \text{in } \Omega$$

as $\varepsilon \downarrow 0$. The other is to associate (1.1) with a differential game and represent the desired solution as its value. In applying these methods the function F is required to be more regular than in the existence results. Thus, in order to get general existence results, one needs additional approximation arguments. See for these approaches G. Barles [1, 2], M. G. Crandall-P.L. Lions [5, 6, 8], L. C. Evans-H. Ishii [9], L. C. Evans-P. E. Souganidis [10], H. Ishii [11, 13], P. E. Souganidis [18], P. L. Lions [15, 16] and references therein.

Our purpose here is to present a new simple, direct method of proving the existence of global solutions of (1.1), which we call Perron's method. The idea is to build a solution as the supremum of subsolutions. This is an analogue for Hamilton-Jacobi equations to the well-known method of finding solutions of the Laplace equation due to O. Perron [17], and it seems also classical in the context of control theory. Our method depends heavily on the notion of (possibly) discontinuous viscosity solution and easily produces such a weak solution of (1.1) under quite general hypotheses. We also present two techniques in Examples 1, 2, and 3 of Section 3 which yield the continuity of the viscosity solution thus obtained. One utilizes the "coercive" structure of (1.1) and the other uses the comparison results for solutions of (1.1).

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