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UNIFORM SOBOLEV INEQUALITIES AND UNIQUE CONTINUATION FOR SECOND ORDER CONSTANT COEFFICIENT DIFFERENTIAL OPERATORS

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1. Introduction. The main goal of this paper is to prove "Sobolev inequalities" for constant coefficient second order differential operators with bounds which do not depend on the lower order terms. More precisely, let $Q(\xi)$ denote a nonsingular real quadratic form on $\mathbb{R}^n$, $n \geq 3$, which, for some $2 \leq j \leq n$, is given by

$$Q(\xi) = -\xi_1^2 - \cdots - \xi_j^2 + \xi_{j+1}^2 + \cdots + \xi_n^2. \tag{1.1}$$

Then we shall prove that, if dual exponents p and p' enjoy the relationship $1/p - 1/p' = 2/n$ (i.e., $p = 2n/(n+2)$, $p' = 2n/(n-2)$), there is an absolute constant C such that whenever $P(D)$ is a constant coefficient operator with complex coefficients and principal part $Q(D)$ one has:

$$\|u\|_{L^{p'}(\mathbb{R}^n)} \leqslant C\|P(D)u\|_{L^p(\mathbb{R}^n)}, \qquad u \in H^{2,p}(\mathbb{R}^n). \tag{1.2}$$

By $H^{2,p}(\mathbb{R}^n)$ we mean the space of functions with two derivatives in $L^p(\mathbb{R}^n)$.

In the special case where $P(D)$ is the Laplace operator, $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$, (1.2) is of course the classical Sobolev inequality. Also, when $P(D)$ is the wave operator, $\square = \partial^2/\partial x_1^2 - \sum_{j=2}^n \partial^2/\partial x_j^2$, (1.2) is due to Strichartz [18]. For elliptic operators special cases of (1.1) related to Carleman inequalities were obtained by the authors [11].

We remark that homogeneity considerations force the exponents p and p' in (1.2) to satisfy the gap condition $1/p - 1/p' = 2/n$. However, as we shall see, for elliptic operators the uniform inequality (1.2) also holds for pairs of exponents r and s which are off of the line of duality, that is:

$$\|u\|_{L^s(\mathbb{R}^n)} \leqslant C\|P(D)u\|_{L^r(\mathbb{R}^n)}, \qquad u \in H^{2,r}(\mathbb{R}^n).$$

Moreover, in this case we shall in fact find necessary and sufficient conditions on the exponents for the above uniform Sobolev inequalities.

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