Vol. 55, No. 1 DUKE MATHEMATICAL JOURNAL © March 1987

THE HOMOLOGY OF A SPACE ON WHICH A REFLECTION GROUP ACTS

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0. Suppose that (W, S) is a Coxeter system, that X is a CW-complex, and that $(X_s)_{s \in S}$ is a family of subcomplexes indexed by S. Given the above data, there is a classical construction of a CW-complex $\mathcal{U}$ with W-action: $\mathcal{U}$ is obtained by pasting together copies of X, one for each element of W. To be more explicit, for each x in X, let W_x denote the subgroup of W generated by the set of s in S such that x belongs to X_s; let $\sim$ denote the equivalence relation on $W \times X$ defined $(w, x) \sim (v, y) \Leftrightarrow x = y$ and $w^{-1}v \in W_x$; the complex $\mathcal{U}$ is then defined as the quotient space $(W \times X)/\sim$. We identify X with the image of $1 \times X$ in $\mathcal{U}$. The subcomplexes wX, $w \in W$, are called the *chambers* of $\mathcal{U}$; while the subcomplexes wX_s, $s \in S$, are the *mirrors* of wX.

The *length* of an element w of W, denoted by $l(w)$, is the smallest integer n such that w is the product of n elements in S. Put

$$S(w) = \{s \in S \mid l(ws) < l(w)\}.$$

(If s is in S, then the element wsw^{-1} acts on $\mathcal{U}$ as a reflection across the mirror wX_s, taking the chamber wX to the adjacent chamber wsX. Therefore, the set $S(w)$ indexes the set of mirrors of wX with the property that the adjacent chamber across the mirror is one chamber closer to X.)

For each subset T of S, let X^T be the subcomplex of X defined by

$$X^T = \bigcup_{t \in T} X_t.$$

THEOREM A. *The homology of $\mathcal{U}$ is isomorphic to the following direct sum,*

$$H_*(\mathcal{U}) \cong \sum_{w \in W} H_*(X, X^{S(w)}). \tag{1}$$

Received March 12, 1985. Partially supported by NSF grant DM58412891.