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# SERRE DUALITY ON COMPLEX SUPERMANIFOLDS

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In 1955 J. P. Serre published his well-known duality theorem (Serre, 1955), which has had many consequences in complex geometry. In particular, as pointed out in his original paper, the classical Riemann-Roch theorem for Riemann surfaces is a very quick consequence. This duality theorem has had many applications and has had a number of generalizations over the past 30 years, and more recently there have been generalizations to the context of supermanifolds (Penkov, 1983), and applications of this in the context of supermanifolds have been discussed in the paper of Ogijevetsky and Penkov (1984). In particular the problems of geometric realizations of representations of super Lie algebras (generalizations of the Bott-Borel-Weil Theorem) are contexts in which this duality theory is useful (see Penkov, 1985).

The recent book of Manin (1984) gives a beautiful introduction to the ideas of supergeometry. This is the study of manifolds with both commuting and anti-commuting coordinates, which was inspired by pioneering work in theoretical physics over the past two decades. In this paper we want to outline a new proof of Serre duality on complex supermanifolds which is a generalization of the original proof of Serre (1955). The proof of Penkov (1983) uses the theory of $\mathscr{D}$-modules, and is a generalization of the proof of Hartshorne (1966) in which he has an extension of Serre duality using the Ext functor. Our proof uses the natural duality between differentiable functions and distributions, and uses the underlying differentiable structure of the complex supermanifold, just as in the original Serre paper. The details will be published elsewhere.

The basic Serre duality theorem includes as a special case the following result. If $V$ is a holomorphic vector bundle on a complex manifold $M$ of complex dimension $n$, then there is a natural duality between $H^q(M, \Omega^p(V))$ and $H^{n-q}(M, \Omega^{n-p}(V^*))$. This result has been generalized by Hartshorne (1966) to a duality between $H^q(M, \mathscr{F})$ and $\operatorname{Ext}^{n-q}(M, \mathscr{F}, \Omega^n)$, where $F$ is a coherent $\mathscr{O}$-module. This is a generalization, since for a locally free $\mathscr{O}$-module $\mathscr{F}$ there is an isomorphism

$$\operatorname{Ext}^k(M, \mathscr{F}, \Omega^n) \cong H^k(M, \Omega^n(\mathscr{F}^*)).$$

Using the theory of $\mathscr{D}$-modules, Serre duality was proved for supermanifolds (Penkov, 1983) in the following form: for a compact complex supermanifold $(M, \mathscr{O})$ of dimension $m|n$, there is a duality between $\operatorname{Ext}^{m-q}(M, \mathscr{F}, \mathscr{B}er_{\mathscr{O}})$ and

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