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ON THE RATIONALITY PROBLEM FOR CONIC BUNDLES

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To Yu. I. Manin on his 50th birthday

Introduction. Let V be a smooth irreducible projective threefold over $\mathbb{C}$. In this paper we are interested in the problem of characterizing rational threefolds V. For V to be rational, it is necessary that $q(V) := h^1(V, \mathcal{O}_V) = 0$ and $\kappa(V) = -\infty$ (κ stands for Kodaira dimension). For surfaces these two properties are also sufficient, but they are not in higher dimensions. However the Reid-Mori program for constructing minimal models (see, for example [11]) and results of Miyaoka [17] imply that every threefold V with $\kappa(V) = -\infty$ is birational to a variety W with at most terminal singularities of one of the following types:

(1) the anticanonical divisor $-K_W$ is ample and $\operatorname{Pic} W \simeq \mathbb{Z}$, i.e., W is a *minimal* $\mathbb{Q}$*-Fano variety*;

(2) There exists a morphism $\delta: W \to C$ onto a smooth curve C whose generic fiber is a del Pezzo surface and $\operatorname{Pic} W \simeq \delta^*\operatorname{Pic} C \oplus \mathbb{Z}$, i.e., W is a *minimal fiber space of del Pezzo surfaces*;

(3) There exists a morphism $\pi: W \to S$ onto a normal surface S, each fiber of π is isomorphic to a conic in $\mathbb{P}^2$, and $\operatorname{Pic} W \simeq \pi^*\operatorname{Pic} S \oplus \mathbb{Z}$, i.e., W is a *minimal conic bundle*. It is known that up to birational equivalence we can restrict ourselves to the case when W and S are nonsingular and projective (see [24], [16], [21]).

Thus the problem of characterizing rational threefolds is divided into three parts, each part requiring a description of rational varieties belonging to one of the three types. We note that if V is rational, then so is C or S in (2) or (3).

In the present paper we study the rationality problem for varieties of the third type, that is, conic bundles. The author's note [4] suggested a conjectural rationality criterion for such varieties, followed by some comments and a sketch of the proof. Here we discuss the problem in more detail.

In §1 we state Conjecture I which gives the rationality criterion for conic bundles, and we prove the sufficiency part (Theorem 1). Necessity is proved in Theorem 2 under the additional assumption that for a rational conic bundle V over S there exists a birational map $\chi: V \dashrightarrow \mathbb{P}^3$ which takes fibers of the morphism $\pi: V \to S$ to conics in $\mathbb{P}^3$. This assumption constitutes Conjecture II. By Theorems 1 and 2, Conjecture II is equivalent to the "only if" part of Conjecture I. We note that Kantor [10] gives a "proof" for Conjecture II, but this

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