Vol. 54, No. 1 DUKE MATHEMATICAL JOURNAL © 1987

CONSTRUCTIVE HIGH-DIMENSIONAL SPHERE PACKINGS

S. N. LITSYN AND M. A. TSFASMAN

To Yu. I. Manin

We use some ideas and methods of algebraic geometry, coding theory, and complexity theory to establish new constructive asymptotic existence bounds for the density of sphere packings (lattice and nonlattice) in Euclidean spaces. The main tools are: on one hand the beautiful construction of A. Bos, J. H. Conway and N. J. A. Sloane, on the other error-correcting codes obtained from algebraic curves. The results of this paper were announced in [1].

CONTENTS

§1. Preliminaries 147
§2. Error-correcting codes 150
§3. Complexity 151
§4. Constructions 154
§5. Decrease of complexity 156
§6. Asymptotically dense packings 157
§7. Number-theoretic lattices 158

§1. Preliminaries. How should we place equal nonintersecting open spheres in $\mathbb{R}^N$ so as to obtain the densest possible packing? A sphere packing is a configuration of nonintersecting equal open spheres in $\mathbb{R}^N$. Let d be the diameter of the spheres; then the distance between any two sphere centers is at least d. Thus a *packing* is a set of points P in $\mathbb{R}^N$ such that the minimum distance between any two is at least d. If this set is an additive subgroup of $\mathbb{R}^N$ it is called a *lattice* or a *lattice packing*. For any packing P the *density* $\Delta(P)$ is defined as the fraction of space covered by spheres. More precisely, let $K_u = \{x = (x_0, \dots, x_n) \mid |x_i| \leqslant u\}$ be a large cube centered at the origin, and $S = S(P)$ the set of points inside the spheres of the packing P; then

$$\Delta(P) = \limsup_{u \to \infty} \mu(S \cap K_u)/\mu(K_u),$$

where μ is the usual measure in $\mathbb{R}^N$ (cf. [2]). Of course $\Delta(P)$ does not depend on the fact that K_u is a cube.

Received November 10, 1986.