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ENDOMORPHISMS AND TORSION OF ABELIAN VARIETIES

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Introduction. Let K be a number field of finite degree over the field $\mathbb{Q}$ of rational numbers, $\bar{K}$ the algebraic closure of K and $G = \mathrm{gal}(\bar{K}/K)$ the Galois group. Let X be an Abelian variety over K and End X the ring of K-endomorphisms of X. The well-known Mordell-Weil theorem [4] asserts that $X(K)$ is a finitely generated commutative group. In particular, the torsion subgroup TORS $X(K)$ of $X(K)$ is finite.

Let p be a prime and K_p the subfield of $\bar{K}$ obtained by adjoining to K all p-power roots of unity in $\bar{K}$. B. Mazur conjectured that $X(K_p)$ is also finitely generated (see [6], [5]). In this connection J.-P. Serre and H. Imai [3] proved (independently) that the torsion subgroup of $X(K_p)$ is finite for each p. Moreover, let K^{cycl} be the compositum of all the K_p, i.e., the field obtained by adjoining to K all roots of unity in $\bar{K}$. K. A. Ribet [8] proved that the torsion subgroup of $X(K^{\mathrm{cycl}})$ is also finite.

Let $K^{\mathrm{ab}} \subset \bar{K}$ be the maximal abelian extension of K. The field K^{ab} contains K^{cycl}; if $K = \mathbb{Q}$, then $\mathbb{Q}^{\mathrm{cycl}} = \mathbb{Q}^{\mathrm{ab}}$ (the Kronecker-Weber theorem).

What can one say about the finiteness of the torsion subgroup TORS $X(K^{\mathrm{ab}})$ of $X(K^{\mathrm{ab}})$? The answer depends on the properties of the following endomorphism algebra of X:

$$\mathrm{End}^\circ X = \mathrm{End}\, X \otimes \mathbb{Q}.$$

Recall ([7], [10], [11]) that X is called of CM-type over K if the finite-dimensional semisimple $\mathbb{Q}$-algebra $\mathrm{End}^\circ X$ contains a semisimple commutative Q-subalgebra (i.e. the finite direct sum of number fields) of dimension $2 \dim X$. If X is a K-simple Abelian variety, then $\mathrm{End}^\circ X$ is a division algebra and the following conditions are equivalent; (1) X is of CM-type over K; (2) $\mathrm{End}^\circ X$ is a number field of degree $2 \dim X$; (3) $\mathrm{End}^\circ X$ contains a number field of degree $2 \dim X$.

If X is of CM-type over K, then all the torsion points of X are defined over K^{ab} [10, 11], i.e., the torsion subgroups of $X(K^{\mathrm{ab}})$ and $X(\bar{K})$ coincide. In particular, TORS $X(K^{\mathrm{ab}})$ is infinite if X is of CM-type over K.

If Y is an Abelian variety defined over K and K-isogenous to X, then TORS $X(K^{\mathrm{ab}})$ is finite if and only if TORS $Y(K^{\mathrm{ab}})$ is finite. If Y is K-isomor-

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