STABLE BUNDLES AND INTEGRABLE SYSTEMS

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§1. Introduction. The moduli spaces of stable vector bundles over a Riemann surface are algebraic varieties of a very special nature. They have been studied for the past twenty years from the point of view of algebraic geometry, number theory and the Yang-Mills equations. We adopt here another viewpoint, considering the *symplectic geometry* of their cotangent bundles. These turn out to be algebraically completely integrable Hamiltonian systems in a very natural way. For rank 2 bundles of odd degree this appeared as a byproduct of an investigation [5] into certain solutions of the self-dual Yang-Mills equations, a subject on which Yuri Manin has had a profound influence.

The cotangent bundle T^*N of an n-dimensional complex manifold N is a completely integrable Hamiltonian system if there exist n functionally independent, Poisson-commuting holomorphic functions on T^*N. These functions for the case where N is the moduli space of stable G-bundles on a compact Riemann surface M, and G a complex semisimple Lie group are easy to describe. The tangent space of the moduli space at a point is identified with the sheaf cohomology group $H^1(M; \mathfrak{g})$ where $\mathfrak{g}$ is a holomorphic bundle of Lie algebras. By Serre duality the cotangent space is $H^0(M; \mathfrak{g} \otimes K)$. An invariant polynomial of degree d on the Lie algebra then gives rise to a map from this cotangent space to the space $H^0(M; K^d)$ of differentials of degree d on M. Taking a basis for the ring of invariant polynomials yields a map to the vector space $W = \oplus_{i=1}^{k} H^0(M; K^{d_i})$ where d_i are the degrees of the basic invariant polynomials. Somewhat miraculously, the dimension of this vector space is always equal to the dimension of the moduli space N, thus providing the n functions.

The Hamiltonian vector fields corresponding to these functions give n commuting vector fields along the fibres of the map to W. The system is called *algebraically* completely integrable if the generic fibre is an open set in an abelian variety and the vector fields are linear. For the system above, at least for the case where G is a classical group, this also turns out to be true, the abelian variety being either a Jacobian or a Prym variety of a curve covering M. The construction of this curve, and its corresponding Jacobian, parallels the solution of differential equations of "spinning top" type involving isospectral deformations of a matrix of polynomials in one variable. A point of the cotangent bundle of the moduli space consists of a stable vector bundle V (with G-structure) and a holomorphic section $\Phi \in H^0(M; \mathfrak{g} \otimes K)$, which gives a holomorphic map Φ: $V \to V \otimes K$. We form the curve of eigenvalues S defined by the equation

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