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TOPOLOGY OF REAL ALGEBRAIC THREEFOLDS

WOJCIECH KUCHARZ

1. Introduction. Let X be a nonsingular d-dimensional algebraic subset of $\mathbf{R}^n$. In this paper $H_*(X, Z/2Z)$ denotes the homology of X built on infinite locally finite chains with coefficients in $Z/2Z$. In particular, if X is compact, then $H_*(X, Z/2Z)$ is the singular homology of X. We address three problems which have attracted the attention of several mathematicians.

Problem 1.1. Let u be a homology class in $H_k(X, Z/2Z)$ which can be represented by an algebraic subset of X. Is it possible to represent u by a nonsingular algebraic subset of X?

If $k = d - 1$, then the answer is "yes" [5], [18]. It is known that every homology class in $H_k(X, Z/2Z)$ can be represented by a C^∞ compact submanifold of X provided that X is compact and $2k + 1 \leqslant d$ [22]. We conjecture that if $2k + 1 \leqslant d$ and a homology class in $H_k(X, Z/2Z)$ can be represented by an algebraic subset of X, then it can be represented by a nonsingular algebraic subset of X. Here we prove this conjecture assuming that $d = 3$ and X is orientable as a C^∞ manifold.

Problem 1.2. Let M be a compact C^∞ k-dimensional submanifold of X. Under what conditions does there exist a C^∞ embedding h: $M \to X$, arbitrarily close to the inclusion map $M \to X$ in the C^∞ topology, such that $h(M)$ is a nonsingular algebraic subset of X?

If there exists a C^∞ embedding h: $M \to X$, sufficiently close to the inclusion map $M \to X$, such that $h(M)$ is a nonsingular algebraic subset of X, then the homology class represented by M in $H_k(X, Z/2Z)$ can also be represented by an algebraic subset of X. The converse is true if X is compact and $k = d - 1$ [5], [18]; we show here that it remains true if X is compact and orientable as a C^∞ manifold, $k = 1$ and $d = 3$ (cf. [2], [4], [13] for other results in this direction).

To each finitely generated projective module over the ring of regular functions on X corresponds, in the standard way, a real vector bundle over X. Vector bundles of this type are called strongly algebraic (cf. Section 2).

Problem 1.3. Characterize strongly algebraic vector bundles among continuous vector bundles over X.

It is known that if ξ is a continuous vector bundle over X which is C^0 isomorphic to a strongly algebraic vector bundle, then for each $k = 0, 1, \ldots, d$,

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