Vol. 53, No. 2 DUKE MATHEMATICAL JOURNAL © June 1986

THE RANGE OF THE TANGENTIAL CAUCHY–RIEMANN OPERATOR

J. J. KOHN

The tangential Cauchy–Riemann operator arises as the restriction of the operator $\bar{\partial}$ to submanifolds of complex manifolds. In this paper we will be mainly concerned with those submanifolds which are boundaries of bounded pseudoconvex domains in $\mathbf{C}^n$, in section 5 (see theorem 5.3) we study the case when $\mathbf{C}^n$ is replaced by a more general class of n-dimensional complex manifolds. We will use the notation introduced in [KR], the restriction of $\bar{\partial}$ will be denoted by $\bar{\partial}_b$. We will prove that the range of $\bar{\partial}_b$ in L_2 is closed. This result, in the case of forms of degree $n-3$, was obtained independently by M. C. Shaw (see [S1]). The case of forms of degree in $n-2$ on boundaries of domains in $\mathbf{C}^n$, was also obtained independently by H. Boas and M. C. Shaw (see [BS]), however their method does not seem to generalize to manifolds as in theorem 5.3. The proof presented here makes extensive use of microlocalizations, of a kind that seems particularly suitable in the study of CR structure (see also [K4]).

For boundaries of strongly pseudo-convex domains the fact that the range of $\bar{\partial}_b$ is closed follows immediately from the result in [KR]. For abstract strongly pseudo-convex compact CR manifolds the result is true only for forms of degree less than or equal to $n-3$, this follows from the subelliptic estimates proved in [K2]. In the case of compact pseudo-convex manifolds of finite ideal type the result holds only for forms of degree less than or equal to $n-3$ and again follows from the subelliptic estimates which are obtained in [K4]. To obtain these subelliptic estimates essential use is made of microlocal methods.

The closed range property underlies all existence and regularity results. For a compact strongly pseudo-convex manifold the closed range property on functions implies that the manifold is embeddable in $\mathbf{C}^m$ (see [K4], this result follows from the regularity estimates in [K4] and the construction of Boutet de Monvel). H. Grauert has constructed compact 3 dimensional, strongly pseudo-convex CR manifolds which are not embeddable. Such examples were also studied by H. Rossi (see [R2]) and by D. Burns (see [B]). It then follows that the range of $\bar{\partial}_b$ is not closed in these examples.

For boundaries of (weakly) pseudo-convex domains it was pointed out by J. P. Rosay (see [R1]) that one can combine the results in [KR] with those in [K1] to prove existence of globally smooth solutions of the equation $\bar{\partial}_b u = f$. This construction of u, appears to "lose" a derivative, and the closed range property is established by showing that in fact u is as regular as f.

Received August 8, 1985.