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PROPER HOLOMORPHIC MAPS FROM BALLS

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1. Introduction and statement of the results. If $\Gamma \subset \mathrm{U}(n)$ is a finite unitary group, the quotient $\mathbf{C}^n/\Gamma$ can be realized as a normal algebraic subvariety V in some $\mathbf{C}^s$ according to a theorem of Cartan [4]. In order to do this we choose a finite number of homogeneous Γ-invariant holomorphic polynomials $q_1, \ldots, q_s$ that generate the algebra of all Γ-invariant polynomials [17]; the induced map $Q = (q_1, \ldots, q_s): \mathbf{C}^n \to \mathbf{C}^s$ is proper and induces a homeomorphism of $\mathbf{C}^n/\Gamma$ onto the image $V = Q(\mathbf{C}^n)$. The restriction of Q to the unit ball $\mathbf{B}^n$ maps the ball properly onto a domain G in V.

Rudin proved a partial converse to this [22]: If $f: \mathbf{B}^n \to G$ is a proper holomorphic map from the ball onto a domain in $\mathbf{C}^n$, $n \geqslant 2$, that extends to a $\mathbf{C}^1$ map on $\overline{\mathbf{B}}^n$, then there are a finite unitary group Γ and an automorphism ϕ of $\mathbf{B}^n$ such that $f = \eta \circ Q \circ \phi$, where $Q: \mathbf{B}^n \to \mathbf{B}^n/\Gamma$ is the quotient projection and $\eta: \mathbf{B}^n/\Gamma \to G$ is a biholomorphic map. The group Γ is generated by reflections, i.e., elements of finite order which fix a complex hyperplane. A result of Bedford and Bell [2] implies the same result even when f does not extend to the closure of $\mathbf{B}^n$; moreover, we may replace G by an arbitrary normal complex space of dimension n. See also [19]. The quotient $\mathbf{C}^n/\Gamma$ is nonsingular if and only if the group Γ is generated by reflections, i.e., elements of finite order in $\mathrm{U}(n)$ that fix a complex hyperplane [12, 20, 22]. The boundary of the image G is never smooth in this case [22].

In this paper we shall study the structure of proper maps from balls into strictly pseudoconvex domains G in complex manifolds. A finite unitary group $\Gamma \subset \mathrm{U}(n)$ is call *fixed point free* if 1 is not the eigenvalue of any $\gamma \in \Gamma \backslash \{1\}$. Equivalently, Γ is fixed point free if it acts without fixed points on $\mathbf{C}^n \backslash \{0\}$.

1.1. THEOREM. *Let $f: \mathbf{B}^n \to G$, $n \geqslant 2$, be a proper holomorphic map into a relatively compact, strictly pseudoconvex domain G in a complex manifold. If f extends to a $\mathbf{C}^1$ map on $\overline{\mathbf{B}}^n$, then there exist a finite fixed point free unitary group Γ and an automorphism ϕ of $\mathbf{B}^n$ such that*

$$f = \eta \circ Q \circ \phi, \tag{1.1}$$

where $Q: \mathbf{B}^n \to \mathbf{B}^n/\Gamma$ is the quotient projection and $\eta: \mathbf{B}^n/\Gamma \to f(\mathbf{B}^n)$ is the normalization of the subvariety $f(\mathbf{B}^n)$ of G.

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