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THE BERGMAN SPACE, THE BLOCH SPACE, AND COMMUTATORS OF MULTIPLICATION OPERATORS

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1. Introduction. Let $\mathbf{D}$ denote the open unit disk in the complex plane $\mathbf{C}$, and let A denote the usual area measure on $\mathbf{C}$. For $1 \leqslant p < \infty$, the Bergman space L_a^p is the Banach space of analytic functions $g: \mathbf{D} \to \mathbf{C}$ such that

$$\|g\|_p = \left[\int_{\mathbf{D}} |g(z)|^p \, dA(z)/\pi\right]^{1/p} < \infty.$$

When $p = 2$, we obtain the Hilbert space L_a^2 with inner product given by

$$\langle f, g \rangle = \int_{\mathbf{D}} f(z)\bar{g}(z)\, dA(z)/\pi.$$

As usual, $H^\infty(\mathbf{D})$ denotes the set of bounded analytic functions on $\mathbf{D}$. For $f \in H^\infty(\mathbf{D})$, the multiplication operator $T_f: L_a^2 \to L_a^2$ is defined by $T_f(g) = fg$. This paper answers the question discussed in [3] of characterizing the functions $f \in H^\infty(\mathbf{D})$ such that $T_f^* T_f - T_f T_f^*$ is a compact operator. I worked on this problem because the multiplication operators on L_a^2 furnish basic examples of subnormal operators, and it is desirable to know as much as possible about them. The theory developed by Brown, Douglas, and Fillmore [6] can be applied to those Hilbert space operators T such that $T^*T - TT^*$ is a compact operator (such operators are called essentially normal).

Let P denote the orthogonal projection of $L^2(\mathbf{D}, dA/\pi)$ onto L_a^2, so $(1 - P)$ is the orthogonal projection of $L^2(\mathbf{D}, dA/\pi)$ onto $(L_a^2)^\perp$. For $f \in L^\infty(\mathbf{D}, dA/\pi)$, the Hankel operator $H_f: L_a^2 \to (L_a^2)^\perp$ is defined by $H_f(g) = (1 - P)(fg)$. An easy calculation (Proposition 3) shows that

$$T_f^* T_f - T_f T_f^* = H_{\bar{f}}^* H_{\bar{f}}$$

for all $f \in H^\infty(\mathbf{D})$. Thus for $f \in H^\infty(\mathbf{D})$, the commutator $T_f^* T_f - T_f T_f^*$ is a compact operator if and only if $H_{\bar{f}}$ is a compact operator; the results in this paper will be stated in terms of $H_{\bar{f}}$ rather than in terms of $T_f^* T_f - T_f T_f^*$.

It is useful (and sometimes more natural) to consider Hankel operators H_f for $f \in L^2(\mathbf{D}, dA/\pi)$ (so f is not necessarily bounded). To do this, we slightly modify the definition of H_f given in the paragraph above by restricting the domain of H_f to $H^\infty(\mathbf{D})$. So now H_f maps $H^\infty(\mathbf{D})$ into $(L_a^2)^\perp$ by the formula $H_f(g) =$

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