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CLOSED TRAJECTORIES FOR QUADRATIC DIFFERENTIALS WITH AN APPLICATION TO BILLIARDS

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Introduction. One aspect of the study of any dynamical system is the problem of periodic orbits. In this paper we consider periodic orbits for the dynamical system of a polygonal table with angles, rational multiples of π, the so called rational billiards. A point in the polygon and a choice of initial angle defines an orbit. Rational billiards were first considered in [F-K] and more recently in [Z-K], [G], [Bosh], [B-K-M], and [K-M-S]. In [B-K-M] the question was raised whether every rational billiard has a periodic orbit. This paper answers that question in the affirmative.

THEOREM 1. *For any rational billiard table there is a dense set of directions each with a periodic orbit.*

Following a line of work begun in [K-M-S] we transform the problem of the dynamics of the billiard flow in the different directions into the problem of considering the flows defined by $\operatorname{Re} e^{i\theta}\phi$ where ϕ is a holomorphic 1-form on a compact Riemann surface and $0 \leqslant \theta < 2\pi$. In fact we prove the more general result.

THEOREM 2. *Let q be any holomorphic quadratic differential on a compact Riemann surface of genus $g \geqslant 2$. There exists a dense set of θ for which $e^{i\theta}q$ has a closed regular vertical trajectory.*

Theorem 2 is more general because billiards only give rise to certain compact Riemann surfaces and holomorphic 1-forms. Further, we consider quadratic differentials whose trajectory structures include those given by 1-forms but also include nonorientable foliations.

For a description of how the orbits on a rational table give rise to the flow of $e^{i\theta}\phi$ on a compact Riemann surface we refer to [B-K-M]. This paper is otherwise independent of that paper.

The idea in the proof of Theorem 2 as in [B-K-M] is to use Teichmüller theory. We will consider all Teichmüller extremal maps defined by $e^{i\theta}q$, for θ in an arbitrary closed interval, a sector in the *Teichmüller disc* and consider this sector projected to the moduli space. We show there must be accumulation points on the boundary of moduli space. By repeating a certain minimizing argument on lengths of curves, we in fact show there must be accumulation points whose

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