Vol. 53, No. 1 DUKE MATHEMATICAL JOURNAL © March 1986

ORBITS OF HOROSPHERICAL FLOWS

S. G. DANI

Given an action of a group G, say, on a locally compact Hausdorff space X (namely, a dynamical system) one would like to understand the closures of the individual orbits and, in particular, identify dense orbits. Thus, for instance, a classical theorem of Kronecker is equivalent to asserting that orbits under (iterates of) a translation of the n-dimensional torus group $\mathrm{T}^n = S^1 \times S^1 \times \cdots \times S^1$ by, say, $a = (e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_n})$ is dense if and only if 1, $\alpha_1, \ldots, \alpha_n$ are linearly independent over Q; in general, the closure is a coset of a closed subgroup. Similar results are known for certain affine automorphisms of tori and analogous systems on nilmanifolds. The reader is referred to [8] for references for the above and a related result, for 'horospherical flows,' involving the study of invariant measures of the action.

Another classical result, due to G. A. Hedlund [12], asserts that every orbit of the horocycle flow associated to a surface of constant negative curvature and finite (Riemannian) area, is either dense or periodic; if the surface is compact then every orbit is dense whereas if the surface is noncompact (but has finite area) then there are precisely as many one-parameter families of periodic orbits as the number of cusps. A horocycle flow, as above, can also be viewed as the action of the one-parameter subgroup $U = \{(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}) \mid t \in \mathrm{R}\}$ of $G = \mathrm{SL}(2, \mathrm{R})$ on the homogeneous space G/Γ, where Γ is an appropriate lattice in G depending on the given surface (a subgroup is called a *lattice* if it is discrete and the corresponding homogeneous space admits a finite invariant measure). The above mentioned result is thus an assertion about the U-action on G/Γ. Observe then that for either possibility for the U-orbit (dense or periodic) there exists a closed subgroup H (namely, $H = G$ or U respectively) such that the closure of the U-orbit is precisely the H-orbit.

Now, more generally, let G be a connected reductive Lie group (e.g., $G = \mathrm{SL}(n, \mathrm{R})$, $n \geqslant 2$) and Γ be a lattice in G. A subgroup U of G is said to be *horospherical* if there exists $g \in G$ such that

$$U = \{u \in G \mid g^j u g^{-j} \to e \text{ as } j \to \infty\}$$

where e is the identity element in G. It is easy to see that $\{(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}) \mid t \in \mathrm{R}\}$ is a horospherical subgroup in $\mathrm{SL}(2, \mathrm{R})$. In analogy to the particular case of Hedlund's theorem when the surface is *compact*, it is well known that if G/Γ is compact then every U-orbit on G/Γ is dense, provided the action of U on G/Γ is

Received January 15, 1985.