Vol. 52, No. 4 DUKE MATHEMATICAL JOURNAL © December 1985

TUBULAR NEIGHBORHOODS IN EUCLIDEAN SPACES

JOSEPH HOWLAND GUTHRIE FU

In the classical study of the geometry of convex bodies $S \subset \mathbf{R}^n$, the *tubular neighborhoods*

$$S_r := \{x \in X : \text{distance}(x, S) \leqslant r\}$$

of S play a crucial role. One of the basic facts in this area is that, even if the boundary of the body S is not smooth, the boundaries of the neighborhoods S_r are hypersurfaces of class $C^{1,1}$ for all $r > 0$. Indeed Federer, in order to generalize the notion of a convex body and to place the techniques of the surrounding geometrical theory in their natural context, introduced the *sets of positive reach* and showed that for any such set S there is $\epsilon = \epsilon(S) > 0$ such that for $0 < r < \epsilon$ the tubular neighborhood S_r has a boundary of class $C^{1,1}$ (cf. [11]). Any convex set S has positive reach, with $\epsilon(S) = \infty$.

The gist both of the classical theory of convex bodies and of Federer's theory is contained in the formula of Steiner, which states that the volumes of the S_r satisfy

$$\text{Vol}(S_r) = \sum_{i=0}^{n} c_i(S) r^i$$

for $r < \epsilon(S)$ and $S \subset \mathbf{R}^n$. Moreover, in case S has a smooth boundary the coefficients $c_i(S)$ are, up to constant factors, the integrals σ_i of the symmetric functions of the principal curvatures of the bounding hypersurface. Exploiting this connection, Minkowski defined the *Quermassintegrale* of a *general* convex body and Federer defined the *curvature measures* of a set of positive reach by means of the c_i, appropriately normalized. For nonsmooth sets S of positive reach, these objects play the roles of the curvature integrals σ_i. Thus, in a very definite sense the boundary of a set of positive reach is "practically C^2."

But of course the tubular neighborhood construction is not without interest even for sets S without *any* given structure. In this connection, Ferry and others (cf. [12], [4], [14]) showed that the smoothing behavior of the construction extends in a weakened sense to its action on completely arbitrary subsets of $\mathbf{R}^2$ and $\mathbf{R}^3$. Specifically, *if $S \subset \mathbf{R}^2$ or $\mathbf{R}^3$ is compact then there is a compact set $C \subset [0, \infty)$ of measure zero such that, if $r \notin C$, then the boundary of S_r is a lipschitz manifold.* Ferry also gave an example to show that the corresponding statement is false in dimensions $\geqslant 4$.

Received March 27, 1985. Revision received July 26, 1985.