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ON IMAGES OF RADON TRANSFORMS

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§0. Introduction. We begin with the X-ray transform R taking functions on $X = \mathbf{R}^3$ to functions on Y, the affine Grassmannian of lines (not necessarily through the origin) in $\mathbf{R}^3$. If $f(x)$ is a smooth compactly supported function on $\mathbf{R}^3$ and ℓ is a line in Y, then

$$Rf(\ell) = \int_{\ell} f.$$

A line ℓ which is not parallel to the xy plane can be parametrized as follows:

$$x(t) = \alpha t + \beta, \qquad y(t) = \gamma t + \delta, \qquad z(t) = t$$

so $(\alpha, \beta, \gamma, \delta)$ are local coordinates for Y and $\dim Y = 4$. Since $\dim X = 3$ it is intuitively clear that there are 'more' functions on Y than on X so R cannot be onto. If f is as above then

$$Rf(\alpha, \beta, \gamma, \delta) = \int_{t \in \mathbf{R}} f(\alpha t + \beta, \gamma t + \delta, t)\, dt$$

and we immediately find that

$$\frac{\partial^2 Rf}{\partial \alpha \partial \delta} - \frac{\partial^2 Rf}{\partial \gamma \partial \beta} = 0$$

(differentiating under the integral sign). In a celebrated paper [12] F. John showed that this P.D.E. characterizes the range of R. His goal was to find global solutions of the ultrahyperbolic P.D.E. in four variables

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 g}{\partial z^2} - \frac{\partial^2 g}{\partial w^2} = 0$$

which is equivalent to our original equation by change of variable. This inspired Guillemin and Sternberg to formulate conditions guaranteeing that the image of an integral operator be characterized by a system of partial (or pseudo) differential equations [8]. In this paper, we shall study the complexified projectized version of the X-ray transform and show that its range is also characterized by a P.D.E. (which resembles F. John's equation in local coordinates).

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