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THE UNIFORMIZATION OF THE COMPLEMENT OF THE MANDELBROT SET

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1. Introduction. Let $p_c(z) = z^2 + c$, and p_c^k be the k'th iterate of p_c. The Mandelbrot set, M, is the set of c's for which p_c^k is bounded. It is known that these are the c's for which the Julia set of p_c^k is connected. In [3], Douady and Hubbard show that M is connected by constructing an analytic bijection from M^c to D^c. (All complements are with respect to the Riemann sphere, $\overline{\mathbf{C}}$.) Their construction does not lend itself to computation. We give an alternative construction which does. We also discuss some properties of the coefficients.

2. Notation and preliminaries. Let $f_k(c) = p_c^{k+1}(0)$. Clearly f_k is a monic polynomial of degree 2^k. It is easy to see that $|f_k(c)| > 2$ implies $|f_{k+1}(c)| > |f_k(c)|$. Let $U_k = \{c \in \overline{\mathbf{C}} : |f_k(c)| > 2\}$. Then $U_k \subset U_{k+1}$, and $\bigcup_{0<k<\infty} U_k = M^c$. If $f_k(c) = 0$ then $p_c^{k+1}(0) = 0$ and 0 is periodic under iterations of p_c. Thus c is in M. The only result we will need about M is that it is connected and simply connected, proved in [3]. We will use D_k to represent the closed disk of radius $2^{1/2^k}$ centered at the origin, and D to mean the disk of radius 1.

3. Construction of the uniformizing map

THEOREM 1. *For each k there exists a unique analytic map $\Phi_k : M^c \mapsto \overline{\mathbf{C}}$ satisfying $[\Phi_k(c)]^{2^k} = f_k(c)$ and $\Phi_k(c) \sim c$ as $c \mapsto \infty$.*

Proof. Since M is simply connected, so is M^c. Also $f_k(c)/c^{2^k}$ is analytic and nonzero on M^c. Hence there is an analytic function $g_k : M^c \mapsto \mathbf{C}$ with $f_k(c) = c^{2^k} e^{g_k(c)}$. We have $e^{g_k(\infty)} = 1$ so we may take $g_k(\infty) = 0$, and this specifies g_k completely. Now let $\Phi_k(c) = c e^{g_k(c)/2^k}$. Then Φ_k has the desired properties. □

THEOREM 2. *When restricted to U_k, Φ_k is one-to-one and onto D_k^c.*

Proof. For $c \in U_k$ we have $|\Phi_k(c)|^{2^k} > 2$, so $\Phi_k(U_k) \subset D_k^c$. Let $c_1, c_2, \ldots$ be a sequence in U_k tending to a point of the boundary of U_k. Then $|f_k(c_1)|$, $|f_k(c_2)|, \ldots$ converges to 2, so $\Phi_k(c_1), \Phi_k(c_2), \ldots$ can have no limit in D_k^c. This shows that Φ_k is a proper map on U_k, and so has a degree. Since $\Phi_k(c) \sim c$ this degree is 1, so Φ_k is one-to-one on U_k.

Now choose $z \in D_k^c$. We wish to show that there is a $c \in U_k$ with $\Phi_k(c) = z$. Let $w = z^{2^k}$ and $z_1, z_2, \ldots, z_{2^k} = z$ be the 2^k'th roots of w. Since Φ_k' is non-zero

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