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LENGTH SPECTRA AS MODULI FOR HYPERBOLIC SURFACES

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Let S be a hyperbolic Riemann surface with its induced complete Riemannian metric of constant curvature -1. The geodesic length spectrum of S, denoted by $Spec(S)$, is the subset of $\mathbf{R} \times \mathbf{Z}$ which contains the ordered pair (x, n) whenever there are n distinct smooth closed geodesics on S of length x. We are interested in understanding to what extent the geometry of S is determined by its geodesic length spectrum.

The original motivation for considering this question arose from the fact that, when S is compact, the geodesic length spectrum and the spectrum of the Laplace–Beltrami operator both carry the same information about S [8].

It is known, originally as a result of examples constructed by M. F. Vignéras [10], that a compact hyperbolic surface is not in general uniquely determined by its geodesic length spectrum. More recently, P. Buser [2] has shown that there exist isospectral nonisometric compact surfaces of genus g for $g = 5$ and for all $g \geqslant 7$.

Here we consider the class of surfaces, none compact, whose fundamental groups are free on two generators. Within the moduli space of such a surface the geodesic length spectrum is shown to determine hyperbolic structure. In particular, it follows that two hyperbolic punctured tori with identical length spectra are isometric.

Let $R(g, m, n)$ denote the space of Riemann surfaces of genus g with m punctures and n infinite area ends [1]. A surface S in $R(g, m, n)$ with $2g - 2 + m + n > 0$ carries a unique complete Riemannian metric of curvature -1 which is defined by the conformal structure on S. Two surfaces are isometric if there is a metric preserving diffeomorphism from one to the other or equivalently if they are either conformally or anticonformally equivalent.

The *Nielsen kernel* $K(S)$ of a surface S is the convex hull of the closed geodesics on S. If S belongs to $R(g, m, n)$ then $K(S)$ has n totally geodesic boundary components; one corresponding to each infinite area end of S. A boundary component of $K(S)$ is called a *boundary geodesic*.

Let $R(1, 0, 1; \ell)$ be the subspace of $R(1, 0, 1)$ consisting of all surfaces having a boundary geodesic of length ℓ.

The main result of this paper is

THEOREM 1. *If two surfaces belonging to one of the spaces* $R(1, 1, 0)$, $R(1, 0, 1; \ell)$ $\ell > 0$, *or* $R(0, m, n)$ *with* $m + n = 3$ *have identical geodesic length spectra then they are isometric.*

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