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ON ANALOGS OF THE REISS RELATION FOR CURVES ON RATIONAL RULED SURFACES

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§1. Introduction. In this paper we study a special case of the problem of characterizing the algebraic subvarieties of a given algebraic variety by local, differential criteria. Historically, the first case of this problem to be considered concerned curves in the plane. If we fix a line, l, in the plane and a degree, d, then it is clear that there are algebraic curves of degree d meeting l in any set of d distinct points of l, and that furthermore, the tangent lines at those d points may also be assigned arbitrarily. However, the second- (and higher-) order behavior of an algebraic curve at its intersections with a line is not arbitrary. In 1837 Michel Reiss showed that if C is an algebraic curve of degree d in $\mathbf{P}^2(\mathbf{R})$ which meets a line l in d distinct points $P_i(l)$ and

$$\kappa_i(l) = \text{curvature of } C \text{ at } P_i(l)$$

$$\theta_i(l) = \text{angle from } l \text{ to } C \text{ at } P_i(l),$$

then the so-called *Reiss relation*:

$$\sum_{i=1}^{d} \frac{\kappa_i(l)}{\sin^3\theta_i(l)} = 0 \tag{1.1}$$

must hold. Modern proofs of this fact may be found in [S] and [G-H1].

Although this metric form of the Reiss relation is valid only for curves in the real plane, it is easily seen that (1.1) is equivalent to a relation between the coefficients in the Taylor expansions of the local equations of C at the points $P_i(l)$. In this other, algebraic form, the Reiss relation actually holds for curves defined over any field.

Perhaps the most interesting fact about the Reiss relation is that it characterizes the algebraic curves in $\mathbf{P}^2$. This is so because the Reiss relation has a converse in the following sense. Let C_i be d analytic arcs meeting a fixed line l_0 transversely in d distinct points and define $P_i(l), \kappa_i(l), \theta_i(l)$ as before for lines l near l_0. If the relation (1.1) holds for all lines l sufficiently near l_0, then the C_i lie on an algebraic curve of degree d.

Modern interest in the Reiss relation, its converse, and its generalizations stems primarily from the work of Sophus Lie on translation surfaces. Lie used the fact

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