Vol. 52, No. 4 DUKE MATHEMATICAL JOURNAL © December 1985

HILBERT TRANSFORMS ALONG CURVES, II: A FLAT CASE

MICHAEL CHRIST

1. Introduction. Let $\gamma : [-1, 1] \to \mathbf{R}^n$ denote a reasonably smooth curve. In a number of contexts operators such as

$$Hf(x) = pv \int_{-1}^{1} f(x - \gamma(t)) t^{-1}\, dt,$$

$$Mf(x) = \sup_{0 < r \leqslant 1} r^{-1} \int_{-r}^{r} |f(x - \gamma(t))|\, dt,$$

and their variants have arisen, and their L^p boundedness has been investigated. For references see the survey article [10] as well as [5] and [2]. The case $p = 2$ remains far better understood than the rest of the range of exponents $1 < p < \infty$. This article deals with a very particular class of curves, for which H is already known to be bounded on L^2 and in fact for all $5/3 < p < 5/2$ [6]; we extend this result to the more natural range $(1, \infty)$. Our method also applies to M, and yields positive results on $L(\log L)^\alpha$ for a certain $\alpha > 0$. An example typical of the class of curves under consideration here is $\gamma(t) = (t, \operatorname{sgn}(t) \cdot \exp(-C|t|^{-q}))$, for some $q, C > 0$.

Our contribution concerns the passage from L^2 estimates to other L^p. For motivation consider the example $\gamma(t) = (t, t^3) \in \mathbf{R}^2$, where H and M are already known to be bounded on L^p for all $p \in (1, \infty)$. H may profitably be viewed as a limiting case of certain better behaved and understood operators, which are given by convolution with principal-value distributions k satisfying $k \in C^\infty(\mathbf{R}^2 \setminus \{0\})$, $k(rx_1, r^3 x_2) \equiv r^{-4} k(x_1, x_2)$ for all $r > 0$, and satisfying an appropriate cancellation condition. The L^p theory for H in [10] and in [2] depends on the more elementary fact that these less singular operators are bounded on L^p, and are of weak type on L^1. For more general curves γ devoid of all homogeneity, our strategy will be to construct such a class of better operators, closely tied to the geometry of the particular γ given. Then we will show that these operators are of weak type on L^1 by emulating the classical theory of Calderón and Zygmund [1]. Once this is accomplished, H itself may be treated through the method of [2], which in turn is a small modification of that of [10]. The reader

Received November 12, 1984. Corrected version received February 25. Research supported by NSF grant DMS-8413451.