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SYMMETRIES AND OTHER TRANSFORMATIONS OF THE COMPLEX MONGE–AMPÈRE EQUATION

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1. Introduction. A transformation of an equation (algebraic or differential) is a transformation that carries solutions of the equation into solutions of a related equation. This is admittedly a vague definition. A less elusive notion is that of a symmetry: this is a transformation carrying solutions of the given equation into solutions of the same equation. The study of symmetries of a given equation is important, because the knowledge of symmetries may lead to the explicit construction of solutions; in other cases it may explain why the solutions cannot be obtained using certain operations. This is of course the main idea behind Galois and Lie theory.

The particular equation we are going to study is the complex Monge–Ampère equation $\det \partial^2 u/\partial \bar{z}_j \partial z_k = 0$ for the real valued function u defined on a subdomain of $\mathbf{C}^n$. It happens that this equation can be given an invariant form

$$(\bar{\partial}\partial u)^n = 0, \tag{1.1}$$

where $\bar{\partial}\partial u = \sum u_{\bar{z}_j z_k} d\bar{z}_j \wedge dz_k$, and the power is exterior power. This allows equation (1.1) to make sense on an arbitrary complex manifold of dimension n.

The (homogeneous) Monge–Ampère equation (1.1) was first introduced in [B], then studied in [C-L-N], [B-T1]. It is of great importance in complex potential theory (see [B-T2]) and value distribution theory (see [G-K]). Related equations are studied in relativity and algebraic geometry.

As implied by the invariant form of (1.1), this equation certainly has holomorphic changes of coordinates as symmetries. We are going to exhibit many more symmetries that are not merely changes of coordinates: indeed, the new coordinates, the new function (and its gradient) are expressed in terms of the old coordinates, the old function and its gradient. In modern language, the transformation acts not so much on the manifold M where the function is defined, but rather on an extended phase space $\mathbf{R} \times T^*M$. The first example of such a transformation is due to Euler. They have been systematically investigated by S. Lie ([Li1]), who termed such transformations tangent transformations. He also considered the problem of how these transformations transformed a class of second order equations for a function of two real variables, the so-called real Monge–Ampère equations (see [Li2]). As far as we can judge, his results have no consequence to the study of the complex Monge–Ampère equation (1.1).

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