Vol. 52, No. 4   DUKE MATHEMATICAL JOURNAL ©   December 1985

HURWITZ FAMILIES AND ARITHMETIC GALOIS GROUPS

KEVIN COOMBES AND DAVID HARBATER

Introduction. This paper is concerned with Galois branched covers of the Riemann sphere. In the first section we consider the moduli problem for such covers, including whether Hurwitz families exist and whether they are universal. In the second section we turn to arithmetic questions, involving finding fields of definition and of moduli for covers and for Hurwitz families. In the third section we discuss some techniques for constructing covers with given groups and field of definition, in special cases.

We wish to thank M. Artin, G. Glauberman, and J. Thompson for helpful discussions regarding material in this paper. We also wish to thank M. Fried, both for discussions regarding this material and for comments on the preliminary draft of this paper.

§1. Hurwitz families. In this section we will consider the moduli problem for Galois branched covers of the Riemann sphere $\mathbf{P}^1_{\mathbf{C}}$. We ask whether there is a coarse moduli space, whether this space has an associated family, and whether this family is universal. We do this by first studying the (easier) case of pointed covers, and then forgetting the base points.

We begin by fixing terminology. By a (branched) *cover* $X \xrightarrow{\pi} Y$ (of normal varieties or complex manifolds) we mean a surjective and generically étale map—e.g., any nonconstant map between Riemann surfaces. In the case of Riemann surfaces, the finite set $L \subset Y$ of points at which π is not étale is the *branch locus* of the cover, and generally we will consider such covers taken together with an ordering $(\xi_1, \ldots, \xi_r)$ of the branch points. A *Galois* covering is one which is connected and whose Galois group $\mathrm{Gal}_Y X$ of covering transformations acts transitively on the fibers. Related to this notion are two others: If G is an (abstract) finite group, a *G-cover* $X \to Y$ is a cover together with an inclusion $i: G \hookrightarrow \mathrm{Gal}_Y X$ whose image acts transitively on the fibers. If X is connected such a G-cover will be called *G-Galois*; any such cover is of course Galois in the above sense, and the inclusion i is an isomorphism. Observe that for a G-Galois cover, each $\gamma \in G$ acts on X in a way that conjugates the G-action by γ; thus the automorphisms of a G-Galois cover are in bijection with the center Z of G.

Received November 7, 1984. Research of first author supported in part by NSF. Research of second author supported in part by NSF and the Sloan Foundation.